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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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The Affine Theory of Electricity and Gravitation

DOMINIC G. B. EDELEN

Communicated by R. A. TOUPIN

Contents

Page
1. Notation 2
2. The Affine Curvature Tensor and Its Contractions 2
3. Variational Considerations and the Affine Field Equations 5
4. Equations of Maxwellian Form as Natural Consequences of the Affine Field Equations 6
5. Hamiltonian Form of the Affine Field Equations 9
6. Conservation Laws 11
7. The Einstein Vacuum Fields as Exact Solutions of the Affine Field Equations in Empty Space 11
8. Semi-Empty Space and the Fields Resulting from Scalar Density Lagrangean Functions 14
9. Classic Vacuum Electrodynamics as Exact Solutions of the Affine Field Equations 17

A general structural theory of fields which arise from variational statements in a non-metric but affinely connected space without torsion is shown to be a natural setting for an intrinsic geometric description of electromagnetic, gravitational, and matter fields. No specific intrinsic geometry beyond that of an affinely connected space is assumed in the derivation of the field equations, and no particular transformation properties are assumed for the Lagrangean function. The field equations which result are thus not necessarily tensorial in nature, although equations of Maxwellian form follow in a natural way together with definite constitutive relations for the charge-current density and the charge-current potential. An electromagnetic field tensor is obtained even if the field equations be nontensorial and is shown to be twice the skewsymmetric part of the third contraction of the affine curvature tensor. This field tensor is fundamental in the development, and it is shown that its vanishing is a necessary condition for the affinely connected space to reduce to a metric space. Under an explicit definition of "emptiness" the general affinely connected space is shown to reduce, for a particular Lagrangean function, to a metric space in which EINSTEIN'S equations for the vacuum gravitational field are satisfied. For a general class of Lagrangean functions the space which is defined as "semi-empty" is shown to be a manifold with a Weyl geometry; and for a particular Lagrangean function of this class the field equations describe classic vacuum electrodynamics. It is then shown that the classic vacuum electrodynamic and gravitational fields are a simple separation of effects.

1. Notation

A point set D contained in a space $\mathscr{E}$ will be referred to as a domain if it is an open, connected, non-empty point set. We denote by D^* the closure of D. By the boundary of D^* shall be meant the set $D^* \ominus D$, where $\ominus$ stands for the set-theoretic difference.

Partial differentiation will be denoted by a comma, and covariant differentiation relative to the affinity $L^\alpha_{\beta\gamma}$ will be denoted by a semi-colon. When a partial derivative is to be taken with respect to an element of a collection of functions defined over D or D^*, it will be denoted by a comma followed by the symbol for the particular element as a subscript ($H_{,\psi} = \partial H / \partial \psi$).

We adopt the following symbolic representations for volume and surface differentials:

$$d V \stackrel{\text{def}}{=} \prod_{i=1}^{4} d x^i,$$

$$N_\gamma \, d S \stackrel{\text{def}}{=} d x^1 \, d x^2 \ldots d x^{\gamma-1} \, d x^{\gamma+1} \ldots .$$

The symmetric and skewsymmetric parts of a tensor $T_{\alpha\beta}$ will be denoted by $T_{(\alpha\beta)}$ and $T_{[\alpha\beta]}$ respectively.

2. The Affine Curvature Tensor and Its Contractions

Let $\mathscr{E}$ be a four-dimensional Hausdorff space into which a structure is introduced by requiring that at every point of $\mathscr{E}$ there be defined a *symmetric* affine connection $L^\alpha_{\beta\gamma}$. The **affine curvature tensor** $K^\eta{}_{\alpha\chi\mu}$, associated with $L^\alpha_{\beta\gamma}$, is defined by

$$K^\eta{}_{\alpha\chi\mu} = L^\eta{}_{\alpha\mu,\chi} - L^\eta{}_{\alpha\chi,\mu} + L^\eta{}_{\sigma\chi} L^\sigma{}_{\alpha\mu} - L^\eta{}_{\sigma\mu} L^\sigma{}_{\alpha\chi}; \tag{2.1}$$

from which

$$A^\eta{}_{;\chi;\mu} - A^\eta{}_{;\mu;\chi} = - K^\eta{}_{\alpha\chi\mu} A^\alpha \tag{2.2}$$

where A^α are the components of a contravariant tensor of rank one and weight zero. We may build three new tensors from the tensor $K^\eta{}_{\alpha\chi\mu}$ by contraction.

The first contraction:

$$A_{\chi\mu} \stackrel{\text{def}}{=} K^\alpha{}_{\alpha\chi\mu} = L^\alpha{}_{\alpha\mu,\chi} - L^\alpha{}_{\alpha\chi,\mu}. \tag{2.3}$$

The second contraction:

$$B_{\alpha\mu} \stackrel{\text{def}}{=} K^\eta{}_{\alpha\eta\mu} = L^\eta{}_{\alpha\eta,\mu} - L^\eta{}_{\eta\alpha,\mu} + L^\beta{}_{\eta\mu} L^\eta{}_{\beta\alpha} - L^\beta{}_{\beta\eta} L^\eta{}_{\alpha\mu}. \tag{2.4}$$

The third contraction:

$$C_{\alpha\chi} \stackrel{\text{def}}{=} K^\eta{}_{\alpha\chi\eta} = L^\eta{}_{\eta\alpha,\chi} - L^\eta{}_{\alpha\chi,\eta} + L^\beta{}_{\eta\chi} L^\eta{}_{\beta\alpha} - L^\beta{}_{\beta\eta} L^\eta{}_{\alpha\chi}. \tag{2.5}$$

The three tensors defined by equations (2.3) through (2.5) are not algebraically independent. Specifically, since the affine connection of $\mathscr{E}$ is assumed to be symmetric, the affine curvature tensor admits the following complete set of algebraic identities[1]:

$$K^\alpha{}_{\zeta(\mu\eta)} = 0, \qquad K^\alpha{}_{[\zeta\mu\eta]} = 0.$$

[1] Thomas, T. Y.: The Differential Invariants of Generalized Spaces, p. 132. Cambridge: Cambridge University Press 1934.

Substituting equations (2.3) through (2.5) into these identities, we have

$$B_{\alpha\mu} = -C_{\alpha\mu}, \qquad A_{\alpha\mu} = -2C_{[\alpha\mu]}.$$

Thus, $C_{\alpha\mu}$ is the only independent contraction of the affine curvature tensor, the other two contractions being obtained from $C_{\alpha\mu}$ by the above equations. Rather than work with the tensor $C_{\alpha\mu}$ directly, we decompose $C_{\alpha\mu}$ into its skew-symmetric and symmetric parts, as defined by

$$A_{\alpha\mu} = -2C_{[\alpha\mu]} = L^{\varepsilon}{}_{\varepsilon\mu,\alpha} - L^{\varepsilon}{}_{\varepsilon\alpha,\mu} \tag{2.6}$$

and

$$R_{\alpha\mu} \overset{\text{def}}{=} C_{(\alpha\mu)} = \tfrac{1}{2}(L^{\varepsilon}{}_{\varepsilon\alpha,\mu} + L^{\varepsilon}{}_{\varepsilon\mu,\alpha}) - L^{\varepsilon}{}_{\alpha\mu,\varepsilon} + L^{\beta}{}_{\varepsilon\mu} L^{\varepsilon}{}_{\beta\alpha} - L^{\beta}{}_{\beta\varepsilon} L^{\varepsilon}{}_{\alpha\mu} \tag{2.7}$$

respectively, and work with these tensors separately. As will be seen in Section 4, this decomposition of $C_{\alpha\mu}$ admits a simple and direct separation of electromagnetic and non-electromagnetic effects.

The vanishing or non-vanishing of the tensor $A_{\alpha\beta}$ determines certain intrinsic properties of the space $\mathscr{E}$ which will be required in succeeding sections.

Theorem 1. *A covariant constant scalar density*[2] *field is possible in* $\mathscr{E}$ *if and only if* $A_{\alpha\beta}=0$.

Proof. By definition, a density $C(\boldsymbol{x})$ forms a covariant constant field in $\mathscr{E}$ if and only if

$$C_{;\gamma} \overset{\text{def}}{=} C_{,\gamma} - L^{\varepsilon}{}_{\varepsilon\gamma} C = 0$$

at all points of $\mathscr{E}$. From the above equation we obtain

$$(\ln C)_{,\gamma} = L^{\varepsilon}{}_{\varepsilon\gamma}.$$

These equations are integrable for $\ln C$, and hence for C, if and only if $L^{\varepsilon}{}_{\varepsilon[\gamma,\mu]}=0$, which implies $A_{\mu\gamma}=0$ by equation (2.6).

Theorem 2. *A covariant constant quadrivector field is possible in* $\mathscr{E}$ *if and only if* $A_{\alpha\beta}=0$.

Proof. The proof, using the integrability theorem for mixed systems of partial differential equations, is like that of Theorem 1[3].

An affinely connected space is said to be a metric space if there is defined at all points a tensor $g_{\alpha\beta}$ satisfying the following postulates:

(i) $$g_{[\alpha\beta]} = 0, \tag{2.8}$$

(ii) $$g = \det(g_{\alpha\beta}) \neq 0, \tag{2.9}$$

(iii) $$g_{\alpha\beta;\gamma} = 0\,[4], \tag{2.10}$$

in which case the fundamental quadratic form is given by $g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$.

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[2] A quantity $C(\boldsymbol{x})$ is said to form a scalar density if, under a transformation of $\boldsymbol{x}$ of class C^1, it transforms according to the law $'C = [\det('\boldsymbol{x}_{,\boldsymbol{x}})]^{-1} C$. This definition is included since the term "scalar density" is used by different authors to mean several different things.

[3] SCHOUTEN, J. A.: Ricci-Calculus, pp. 154—155. Berlin-Göttingen-Heidelberg: Springer 1954.

[4] SCHOUTEN: *op. cit.*, p. 132.

Theorem 3. *The affinely connected space $\mathscr{E}$ cannot reduce to a metric space if $A_{\alpha\beta} \neq 0$.*

Proof. It can be shown that a necessary condition for the existence of a tensor field $g_{\alpha\beta}$ satisfying the postulates of a metric space is expressed by the equations [5]

$$g_{\varepsilon\beta} K^{\varepsilon}{}_{\alpha\gamma\delta} + g_{\alpha\varepsilon} K^{\varepsilon}{}_{\beta\gamma\delta} = 0.$$

Multiplying by $g^{\lambda\beta}$ and contracting with respect to (λ, α) yields $2K^{\lambda}{}_{\lambda\gamma\delta} = 0$. This implies, by equation (2.3), that $A_{\gamma\delta} = 0$, from which the theorem follows.

Let there be defined over an affinely connected space a tensor field $g_{\alpha\beta}$ satisfying the following postulates:

(i) $$g_{[\alpha\beta]} = 0, \tag{2.11}$$

(ii) $$g = \det(g_{\alpha\beta}) \neq 0, \tag{2.12}$$

(iii) $$g_{\alpha\beta;\gamma} = -Q_{\gamma\alpha\beta}, \tag{2.13}$$

where (i) and (iii) imply

$$Q_{\gamma[\alpha\beta]} = 0. \tag{2.14}$$

Under these postulates it can be shown[6] that a symmetric affine connection $L^{\alpha}{}_{\mu\lambda}$ is uniquely determined and has the form

$$L^{\alpha}{}_{\mu\lambda} = \Gamma^{\alpha}{}_{\mu\lambda} + \tfrac{1}{2} g^{\alpha\varepsilon}(Q_{\mu\lambda\varepsilon} + Q_{\lambda\varepsilon\mu} - Q_{\varepsilon\mu\lambda}), \tag{2.15}$$

where $\Gamma^{\alpha}{}_{\mu\lambda}$ are the Christoffel symbols based on $g_{\alpha\beta}$, *i.e.*,

$$\Gamma^{\alpha}{}_{\mu\lambda} = g^{\alpha\varepsilon}(g_{\lambda\varepsilon,\mu} + g_{\mu\varepsilon,\lambda} - g_{\mu\lambda,\varepsilon})/2 \tag{2.16}$$

and

$$\Gamma^{\alpha}{}_{\alpha\lambda} = (\ln|g|)_{,\lambda}. \tag{2.17}$$

From equation (2.15) we have

$$L^{\alpha}_{[\mu\lambda]} = \tfrac{1}{2} g^{\alpha\varepsilon}(Q_{[\mu\lambda]\varepsilon} + Q_{[\lambda|\varepsilon|\mu]})$$

upon using equation (2.14). Thus $L^{\alpha}{}_{\mu\lambda}$ is symmetric if and only if

$$Q_{[\mu\lambda]\varepsilon} + Q_{[\lambda|\varepsilon|\mu]} = 0.$$

Rearranging the terms in the last equation gives

$$Q_{\mu[\lambda\varepsilon]} = Q_{\lambda[\mu\varepsilon]},$$

which is identically satisfied since both terms vanish by equation (2.14). Thus equation (2.15) defines a symmetric affine connection. Set

$$T_{\mu\lambda\varrho} = \tfrac{1}{2}(Q_{\mu\lambda\varrho} - Q_{\lambda\varrho\mu} + Q_{\varrho\mu\lambda}), \tag{2.18}$$

and denote covariant differentiation relative to the connection $\Gamma^{\alpha}{}_{\beta\gamma}$ by ∇; then[7]

$$\begin{aligned} R_{\mu\lambda} = J_{\mu\lambda} + \nabla_{\zeta} g^{\varrho\zeta} T_{\mu\lambda\varrho} - \tfrac{1}{2}\nabla_{\mu} g^{\varrho\zeta} T_{\zeta\lambda\varrho} - \tfrac{1}{2}\nabla_{\lambda} g^{\varrho\zeta} T_{\zeta\mu\varrho} + \\ + g^{\varrho\zeta} g^{\varepsilon\tau} T_{\tau\varrho\varepsilon} T_{\mu\lambda\zeta} - g^{\zeta\varrho} g^{\varepsilon\tau} T_{\mu\varrho\varepsilon} T_{\tau\lambda\zeta}, \end{aligned} \tag{2.19}$$

$$A_{\tau\mu} = -2(g^{\varrho\zeta} T_{[\mu|\zeta\varrho|})_{,\tau]} \tag{2.20}$$

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[5] Thomas, T. Y.: *op. cit.*, pp. 218–219. (The $B^{\varepsilon}{}_{\alpha\gamma\delta}$ of Thomas is the same as the negative of our $K^{\varepsilon}{}_{\alpha\gamma\delta}$).

[6] Schouten, J. A.: *op. cit.*, p. 132, equation 3.5.

[7] Schouten, J. A.: *op. cit.*, p. 141.

where $J_{\alpha\beta}$ is the third contraction of the affine curvature tensor associated with the connection $\Gamma^{\alpha}_{\ \beta\gamma}$.

Let $\delta L^{\alpha}_{\ \beta\gamma}$ be an arbitrary variation of $L^{\alpha}_{\ \beta\gamma}$ which is constrained by the condition that the symmetry of $L^{\alpha}_{\ \beta\gamma}$ shall be preserved under the process of variation, *i.e.*, $\delta L^{\alpha}_{\ \beta\gamma} = \delta L^{\alpha}_{\ \gamma\beta}$. We may easily show, by direct computation, that the following result holds:

$$\delta K^{\alpha}_{\ \beta\gamma\mu} = (\delta L^{\alpha}_{\ \beta\mu})_{;\gamma} - (\delta L^{\alpha}_{\ \beta\gamma})_{;\mu},$$

from which we obtain

$$\delta A_{\alpha\beta} = (\delta L^{\varepsilon}_{\ \varepsilon\beta})_{;\alpha} - (\delta L^{\varepsilon}_{\ \varepsilon\alpha})_{;\beta}, \tag{2.21}$$

$$\delta R_{\alpha\beta} = \tfrac{1}{2}(\delta L^{\varepsilon}_{\ \varepsilon\alpha})_{;\beta} + \tfrac{1}{2}(\delta L^{\varepsilon}_{\ \varepsilon\beta})_{;\alpha} - (\delta L^{\varepsilon}_{\ \alpha\beta})_{;\varepsilon}. \tag{2.22}$$

3. Variational Considerations and the Affine Field Equations

Let $Q_{(\varrho)}$ be a collection of functions, which we shall call the **exterior matter fields;** their transformation properties are left unspecified. Let $\mathscr{L}(A_{\alpha\beta}, R_{\alpha\beta}, L^{\alpha}_{\ \beta\gamma}, Q_{(\varrho)}, Q_{(\varrho),\varepsilon}, x)$ be a function whose transformation properties are unspecified and which is of class C^2 in its arguments; this function will be called the Lagrangean. Although we could also include $L^{\alpha}_{\ \beta\gamma,\varepsilon}$ as an explicit argument of $\mathscr{L}$ in addition to its implicit occurrence through $A_{\alpha\beta}$ and $R_{\alpha\beta}$, this additional complexity is found unnecessary. We wish to determine $L^{\alpha}_{\ \beta\gamma}$ and $Q_{(\varrho)}$ such that

$$\delta \int_{D^*} \mathscr{L}\, dV = 0 \tag{3.1}$$

for all variations $\delta L^{\alpha}_{\ \beta\gamma}$ and $\delta Q_{(\varrho)}$ which vanish on the boundary of D^* and are such that $\delta L^{\alpha}_{[\beta\zeta]} = 0$.

Define the quantities $I^{\alpha\beta}$ and $H^{\alpha\beta}$ by

$$I^{\alpha\beta} = \mathscr{L}_{,R_{\alpha\beta}}, \tag{3.2}$$

$$H^{\alpha\beta} = \mathscr{L}_{,A_{\alpha\beta}} \tag{3.3}$$

which are symmetric and skewsymmetric respectively, due to the respective symmetry and skewsymmetry of $R_{\alpha\beta}$ and $A_{\alpha\beta}$. Using equations (2.21) and (2.22), we obtain, by direct computation,

$$\begin{aligned}
\delta \int_{D^*} \mathscr{L}\, dV = & \int_{D^*} [\mathscr{L}_{,Q_{(\varrho)}} - (\mathscr{L}_{,Q_{(\varrho),\zeta}})_{,\zeta}]\, \delta Q_{(\varrho)}\, dV + \\
& + \int_{D^* \ominus D} [H^{\alpha\beta}(\delta L^{\sigma}_{\ \sigma\beta} N_\alpha - \delta L^{\sigma}_{\ \sigma\alpha} N_\beta) + (\mathscr{L}_{,Q_{(\varrho),\zeta}})\, \delta Q_{(\varrho)} N_\zeta + \\
& \qquad + I^{\alpha\beta}(\tfrac{1}{2}\, \delta L^{\varepsilon}_{\ \varepsilon\beta} N_\alpha + \tfrac{1}{2}\, \delta L^{\varepsilon}_{\ \varepsilon\alpha} N_\beta - \delta L^{\varepsilon}_{\ \alpha\beta} N_\varepsilon)]\, dS + \\
& + \int_{D^*} [- I^{\beta\varepsilon}_{\ \ ,\varepsilon}\, \delta^{\gamma}_{\alpha} + I^{\beta\gamma}_{\ \ ,\alpha} - I^{\beta\gamma} L^{\varepsilon}_{\ \varepsilon\alpha} + I^{\varepsilon\gamma} L^{\beta}_{\ \varepsilon\alpha} + \\
& \qquad + I^{\varepsilon\beta} L^{\gamma}_{\ \varepsilon\alpha} - I^{\varepsilon\varrho} L^{\beta}_{\ \varepsilon\varrho}\, \delta^{\gamma}_{\alpha} + 2 H^{\gamma\varepsilon}_{\ \ ,\varepsilon}\, \delta^{\beta}_{\alpha} + \partial_{L^{\alpha}_{\ \beta\gamma}} \mathscr{L}]\, \delta L^{\alpha}_{\ \beta\gamma}\, dV
\end{aligned}$$

where $\partial_{L^{\alpha}_{\ \beta\gamma}} \mathscr{L}$ is evaluated holding $A_{\alpha\beta}$ and $R_{\alpha\beta}$ constant. This result can be simplified significantly by interpreting $I^{\alpha\beta}$ *as if* it were a contravariant tensor of rank two and weight one and then applying the operation of covariant differentiation with respect to $L^{\alpha}_{\ \beta\gamma}$, *i.e.*,

$$I^{\alpha\beta}_{\ \ :\zeta} \overset{\text{def}}{=} I^{\alpha\beta}_{\ \ ,\zeta} + L^{\alpha}_{\ \varepsilon\zeta} I^{\varepsilon\beta} + L^{\beta}_{\ \varepsilon\zeta} I^{\varepsilon\alpha} - L^{\varepsilon}_{\ \varepsilon\gamma} I^{\alpha\beta}.$$

It must be noted that $I^{\alpha\beta}$ is not necessarily a tensor density; it will be one only if $\mathscr{L}$ is a scalar density, but this is not assumed at present. With this operational agreement, we have

$$\begin{aligned}\delta \int_{D^*} \mathscr{L}\, dV = \int_{D^*} [\mathscr{L}_{,Q_{(\varrho)}} - (\mathscr{L}_{,Q_{(\varrho)},\zeta})_{,\zeta}]\, \delta Q_{(\varrho)}\, dV + \\ + \int_{D^*} [\partial_{L^\alpha_{\beta\gamma}} \mathscr{L} + I^{\beta\gamma}{}_{:\alpha} - \delta^\gamma_\alpha I^{\beta\varepsilon}{}_{:\varepsilon} + 2\delta^\beta_\alpha H^{\gamma\varepsilon}{}_{,\varepsilon}]\, \delta L^\alpha_{\beta\gamma}\, dV + \\ + \int_{D^*\ominus D} [H^{\alpha\beta}(\delta L^\varepsilon{}_{\varepsilon\beta} N_\alpha - \delta L^\varepsilon{}_{\varepsilon\alpha} N_\beta) + \\ + I^{\alpha\beta}(\tfrac{1}{2}\delta L^\varepsilon{}_{\varepsilon\beta} N_\alpha + \tfrac{1}{2}\delta L^\varepsilon{}_{\varepsilon\alpha} N_\beta - \delta L^\varepsilon{}_{\alpha\beta} N_\varepsilon) + \mathscr{L}_{,Q_{(\varrho)},\zeta}\, \delta Q_{(\varrho)} N_\zeta]\, dS.\end{aligned} \tag{3.4}$$

Requiring the variations to vanish on the boundary of D^* and $\delta L^\alpha_{[\beta\gamma]} = 0$, we obtain the following necessary conditions for (3.1) to hold:

$$2I^{\beta\gamma}{}_{:\alpha} - \delta^\gamma_\alpha I^{\beta\varepsilon}{}_{:\varepsilon} - \delta^\beta_\alpha I^{\gamma\varepsilon}{}_{:\varepsilon} + 2\partial_{L^\alpha_{\beta\gamma}} \mathscr{L} + 2\delta^\beta_\alpha H^{\gamma\varepsilon}{}_{,\varepsilon} + 2\delta^\gamma_\alpha H^{\beta\varepsilon}{}_{,\varepsilon} = 0, \tag{3.5}$$

$$\mathscr{L}_{,Q_{(\varrho)}} - (\mathscr{L}_{,Q_{(\varrho)},\zeta})_{,\zeta} = 0. \tag{3.6}$$

Equations (3.5) and (3.6) are $40 + (\varrho)$ equations for the determination of the $40 + (\varrho)$ field variables[8] $L^\alpha_{\beta\gamma}$ and $Q_{(\varrho)}$, and are the basic **affine field equations** of this theory.

Theorem 4. *Equation* (3.5) *is equivalent to the equations*

$$H^{\alpha\varepsilon}{}_{,\varepsilon} = \tfrac{1}{10}(3I^{\alpha\varepsilon}{}_{:\varepsilon} - 2\partial_{L^\varepsilon{}_{\varepsilon\alpha}} \mathscr{L}), \tag{3.7}$$

$$(I^{\beta\gamma}\delta^\varepsilon_\alpha - \tfrac{1}{5} I^{\beta\varepsilon}\delta^\gamma_\alpha - \tfrac{1}{5} I^{\gamma\varepsilon}\delta^\beta_\alpha)_{:\varepsilon} = (\tfrac{1}{5}\delta^\beta_\alpha \partial_{L^\varepsilon{}_{\varepsilon\gamma}} + \tfrac{1}{5}\delta^\gamma_\alpha \partial_{L^\varepsilon{}_{\varepsilon\beta}} - \partial_{L^\alpha_{\beta\gamma}}) \mathscr{L}. \tag{3.8}$$

Proof. Contracting equation (3.5) with respect either to (α, β) or to (α, γ) gives equation (3.7). Eliminating $H^{\alpha\beta}$ between equations (3.5) and (3.7) yields equation (3.8).

Note. If we had not assumed that the torsion tensor of $\mathscr{E}$ vanishes, then contracting the equation which would replace equation (3.5) with respect to (α, β) and (α, γ) would give two different right-hand sides to equation (3.7). Thus the resulting equation (3.5) would be replaced by the system composed of equations (3.7), (3.8), and the equation which would result from equating the two resulting right-hand sides of equation (3.7). Such conditions would yield an additional four equations. In this case we should then have $68 + (\varrho)$ equations to determine $64 + (\varrho)$ variables.

4. Equations of Maxwellian Form as Natural Consequences of the Affine Field Equations

In tensorial form, MAXWELL's equations for a linear medium are

$$f_{(\alpha\beta)} = 0, \qquad f_{[\alpha\beta,\gamma]} = 0,$$
$$F^{(\alpha\beta)} = 0, \qquad F^{\alpha\varepsilon}{}_{,\varepsilon} = S^\alpha,$$
$$F^{\alpha\beta} = G^{\alpha\beta\gamma\delta} f_{\gamma\delta},$$

[8] $H^{\alpha\beta}$ and $I^{\alpha\beta}$ are known functions of $L^\alpha_{\beta\zeta}$, $Q_{(\varrho)}$ and their derivatives once the Lagrangean function $\mathscr{L}$ has been given (*cf.* equations (3.2) and (3.3)). In this sense the Lagrangean function plays the role of a constitutive function.

where $G^{\alpha\beta\gamma\delta}$ is a tensor representing the properties of the medium. More generally, we shall say that a system is of **Maxwellian form** if it may be written as

$$f_{(\alpha\beta)}=0, \quad f_{[\alpha\beta,\gamma]}=0,$$
$$F^{(\alpha\beta)}=0, \quad F^{\alpha\varepsilon}{}_{,\varepsilon}=S^{\alpha}, \tag{4.1}$$
$$F^{\alpha\beta}=J(f_{\alpha\beta},\ldots)_{,f_{\alpha\beta}}.$$

We shall refer to $J(f_{\alpha\beta},\ldots)$ as the **constitutive function for the Maxwellian field.**

The tensor $A_{\alpha\beta}$ satisfies the conditions

$$A_{(\alpha\beta)}=0, \quad A_{[\alpha\beta,\gamma]}=0,$$

as is evident from the defining equation (3.6). We have seen that upon contracting the affine field equations we obtain

$$H^{\alpha\varepsilon}{}_{,\varepsilon}=\tfrac{1}{10}(3I^{\alpha\varepsilon}{}_{:\varepsilon}-2\partial_{L^{\varepsilon}{}_{\varepsilon\alpha}}\mathscr{L})$$

(*cf.* Theorem 4). Setting

$$S^{\alpha}=\tfrac{1}{10}(3I^{\alpha\varepsilon}{}_{:\varepsilon}-2\partial_{L^{\varepsilon}{}_{\varepsilon\alpha}}\mathscr{L}), \tag{4.2}$$

we have

$$H^{\alpha\varepsilon}{}_{,\varepsilon}=S^{\alpha}.$$

Combining this result with equation (4.2), and noting that we also have $H^{(\alpha\beta)}=0$, we have proved the following

Theorem 5. *The affine field equations naturally admit the following system of equations of Maxwellian form:*

$$A_{(\alpha\beta)}=0, \quad A_{[\alpha\beta,\gamma]}=0, \tag{4.3}$$
$$H^{(\alpha\beta)}=0, \quad H^{\alpha\varepsilon}{}_{,\varepsilon}=S^{\alpha}, \tag{4.4}$$

where the Lagrangean function $\mathscr{L}$ is the constitutive function for this Maxwellian field.

We interpret (4.2) as the constitutive equation for the **exterior Maxwellian current** S^{α}, and we refer to equations (4.3) and (4.4) as the **affine Maxwellian equations.**

Since $A_{\alpha\beta}$ is a tensor, equations (4.3) insure the existence of a vector f_{μ} such that

$$f_{[\alpha,\beta]}=A_{\alpha\beta}. \tag{4.5}$$

The vector f_{μ}, so defined, will be referred to as the **affine vector potential.** Expanding equation (4.5) by means of equation (3.6), we have

$$f_{[\alpha,\beta]}=2L^{\varepsilon}{}_{\varepsilon[\beta,\alpha]}.$$

Although the quantities $L^{\varepsilon}{}_{\varepsilon\alpha}$ are not components of a vector under general transformations, they do transform as components of a vector under the subgroup of linear transformations. Hence, under the linear group we may take $L^{\varepsilon}{}_{\varepsilon\alpha}$ as the affine vector potential. Under general transformations, we shall consider $L^{\varepsilon}{}_{\varepsilon\alpha}$ as the **components of potential** of the Maxwellian field tensor $A_{\alpha\beta}$. In many respects, treating $L^{\varepsilon}{}_{\varepsilon\alpha}$ as the potential for $A_{\alpha\beta}$ is more fundamental than

considering a vector potential for $A_{\alpha\beta}$. This is due to the fact that we can find coordinate systems in which $L^{\varepsilon}_{\varepsilon\alpha}$ will vanish at a preassigned point (*i.e.*, a normal coordinate system) but not vanish for other choices of coordinate systems[9]. The freedom of choosing $L^{\varepsilon}_{\varepsilon\alpha}$ to vanish at a preassigned point is equivalent to the freedom of choice of a reference value for the potential.

Equation (4.4) yields, by direct computation, the topological conservation law for the exterior Maxwellian current

$$S^{\alpha}_{,\alpha} = 0. \tag{4.6}$$

If $\mathscr{L}$ be a scalar density, then $H^{\alpha\beta}$ will be a contravariant tensor density, and the above topological conservation law becomes the tensorial conservation law $S^{\alpha}_{;\alpha} = 0$.

We have obtained results having the formal structure of MAXWELL'S electrodynamics by a simple separation of the effects of the tensor $A_{\alpha\beta}$ and the tensor $R_{\alpha\beta}$ (*cf.* Theorem 4) without any assumptions about the form of the constitutive equations which describe the physical fields. This is as it should be. If we are to obtain representations for electromagnetic phenomena from a field theory, the basic field equations representative of electromagnetic phenomena (namely equations of Maxwellian form) should turn out to hold independently of the particular field which we wish to investigate; that is to say, independently of the particular constitutive equations used. We must, however, pick the right constitutive equations if theoretical predictions and experimental results are to coincide. Since the constitutive function for the affine Maxwellian field has been shown to be the Lagrangean function, we are faced with the problem which pervades modern physics, namely, that of finding the appropriate Lagrangean functions for the field equations.

Theorem 6. *If* $\mathscr{E}$ *is assumed* a priori *to be a metric space,* i.e., *there exists a tensor* $g_{\alpha\beta}$ *at all points of* $\mathscr{E}$ *which satisfies the postulates* (2.8) *through* (2.10), *then it is impossible to obtain equations of Maxwellian form from a variational principle whose Lagrangean function depends only on contractions of the curvature tensor, the tensor* $g_{\alpha\beta}$ *and its first derivatives, and the affine connection; rather, electromagnetic phenomena must be introduced by the inclusion of additional field variables.*

Proof. By Theorem 3 we have that $A_{\alpha\beta} = 0$ since, by hypothesis, $\mathscr{E}$ is a metric space. Thus all the arguments of the Lagrangean function as hypothesized are symmetric, since $C_{[\alpha\beta]} = \frac{1}{2} A_{\beta\alpha} = 0$. There is thus no skewsymmetric tensor available to replace the tensor $A_{\alpha\beta}$, and hence additional skewsymmetric field variables must be introduced into the Lagrangean function in order for a variational principle to lead to field equations of Maxwellian form.

The variational theory of LANCZOS[10] might appear to contradict Theorem 6 on first reading. This is not the case, however, if one notes that the vector φ_{μ} introduced by LANCZOS through a "canonical transformation" adds to the Lagrangean function of his theory just the skewsymmetric tensor required to yield equations of Maxwellian form. The result stated in Theorem 6 lies at the

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[9] THOMAS, T. Y.: *op. cit.*, Chapter V.

[10] LANCZOS, C.: Electricity and General Relativity. Rev. Mod. Phys. **29**, No. 3 (July 1957).

heart of the problem of obtaining a field theory which naturally admits electromagnetic and gravitational phenomena from a common geometric base. If we wish to retain the assumption of a metric, the problem of describing electromagnetic and gravitational phenomena from a common geometric base becomes formidable. The works of EINSTEIN, SCHROEDINGER and HLAVATÝ[11] presume an affine space having non-vanishing torsion, with its additional complexity, in order to represent gravitation and electromagnetism from a unified metric standpoint.

5. Hamiltonian Form of the Affine Field Equations

The study of systems of partial differential equations is significantly simplified in many cases if the equations are written in Hamiltonian[12] form. Although it is possible to obtain equations in Hamiltonian form by any of several methods, the most direct and useful for our purposes is the method of auxiliary variables introduced by BATEMAN and later by LANCZOS[13].

The basic Lagrangean introduced in the preceding development was assumed to have the following functional form:

$$\mathscr{L} = \mathscr{L}(A_{\alpha\beta}, R_{\alpha\beta}, L^{\alpha}_{\beta\gamma}, Q_{(\varrho)}, Q_{(\varrho),\gamma}, \boldsymbol{x}). \tag{5.1}$$

If we replace $A_{\alpha\beta}$, $R_{\alpha\beta}$ and $Q_{(\varrho),\gamma}$ by the auxiliary variables $a_{\alpha\beta}$, $r_{\alpha\beta}$ and $q_{(\varrho)\gamma}$, so that

$$\mathscr{L} = \mathscr{L}(a_{\alpha\beta}, r_{\alpha\beta}, L^{\alpha}_{\beta\gamma}, Q_{(\varrho)}, q_{(\varrho)\gamma}, \boldsymbol{x}), \tag{5.2}$$

then $a_{\alpha\beta}$, $r_{\alpha\beta}$, $L^{\alpha}_{\beta\gamma}$, $Q_{(\varrho)}$, and $q_{(\varrho)\gamma}$ may be considered as independent provided the following equations of constraint are satisfied:

$$0 = A_{\alpha\beta} - a_{\alpha\beta} = L^{\varepsilon}{}_{\varepsilon\beta,\alpha} - L^{\varepsilon}{}_{\varepsilon\alpha,\beta} - a_{\alpha\beta}, \tag{5.3}$$

$$0 = R_{\alpha\beta} - r_{\alpha\beta} = \tfrac{1}{2}(L^{\varepsilon}{}_{\varepsilon\alpha,\beta} + L^{\varepsilon}{}_{\varepsilon\beta,\alpha}) - L^{\varepsilon}{}_{\alpha\beta,\varepsilon} + L^{\mu}{}_{\varepsilon\alpha} L^{\varepsilon}{}_{\mu\beta} - L^{\mu}{}_{\mu\varepsilon} L^{\varepsilon}{}_{\alpha\beta} - r_{\alpha\beta}, \tag{5.4}$$

$$0 = q_{(\varrho)\zeta} - Q_{(\varrho),\zeta}. \tag{5.5}$$

Multiplying the equations of constraint by the Lagrangean undetermined multipliers $H^{\alpha\beta}$, $I^{\alpha\beta}$ and $W^{(\varrho)\zeta}$ respectively, we obtain the free variational problem

$$\delta \int_{D^*} \mathscr{L}^* \, dV = 0 \tag{5.6}$$

in the variables $a_{\alpha\beta}$, $H^{\alpha\beta}$, $r_{\alpha\beta}$, $I^{\alpha\beta}$, $L^{\alpha}_{\beta\gamma}$, $Q_{(\varrho)}$, $q_{(\varrho)\zeta}$ and $W^{(\varrho)\zeta}$, where

$$\mathscr{L}^* = H^{\alpha\beta}(L^{\varepsilon}{}_{\varepsilon\beta,\alpha} - L^{\varepsilon}{}_{\varepsilon\alpha,\beta}) + Q_{(\varrho),\zeta} W^{(\varrho)\zeta} - \mathscr{H} + \tfrac{1}{2}(L^{\varepsilon}{}_{\varepsilon\alpha,\beta} + L^{\varepsilon}{}_{\varepsilon\beta,\alpha}) I^{\alpha\beta} - L^{\varepsilon}{}_{\alpha\beta,\varepsilon} I^{\alpha\beta} \tag{5.7}$$

and

$$\begin{aligned}\mathscr{H} = a_{\alpha\beta} H^{\alpha\beta} + (L^{\mu}{}_{\mu\varepsilon} L^{\varepsilon}{}_{\alpha\beta} - L^{\mu}{}_{\varepsilon\beta} L^{\varepsilon}{}_{\mu\alpha} + r_{\alpha\beta}) I^{\alpha\beta} + \\ + q_{(\varrho)\zeta} W^{(\varrho)\zeta} - \mathscr{L}(a_{\alpha\beta}, r_{\alpha\beta}, L^{\alpha}_{\beta\gamma}, \ldots).\end{aligned} \tag{5.8}$$

[11] For a brief summary see SCHROEDINGER, E.: The Final Affine Field Laws. II. Proc. R. I. A. **51**, Sect. A. An extensive and detailed discussion can be found in LICHNEROWICZ, A.: Théories Relativistes de la Gravitation et de L'Électromagnétisme. Paris: Masson & Cie. 1955.

[12] EDELEN, D. G. B.: The Invariance Group for Hamiltonian Systems of Partial Differential Equations I. Analysis. Arch. Rational Mech. Anal. **5** (1960).

[13] BATEMAN, H.: Phys. Rev. **38**, 815—819 (1931). LANCZOS, C.: *op. cit.*

Evaluating the variation of $\int\limits_{D^*} \mathscr{L}^* \, dV$ with respect to $a_{\alpha\beta}$, $r_{\alpha\beta}$, $q_{(\varrho)\zeta}$, respectively, and equating the results to zero in accordance with equation (5.6) yields

$$H^{\alpha\beta} = \mathscr{L}_{,a_{\alpha\beta}}, \qquad I^{\alpha\beta} = \mathscr{L}_{,r_{\alpha\beta}}, \qquad W^{(\varrho)\zeta} = \mathscr{L}_{,q_{(\varrho)\zeta}}. \tag{5.9}$$

Comparing equations (5.9) with equations (3.2), (3.3) and noting that $\mathscr{L}_{,A_{\alpha\beta}} = \mathscr{L}_{,a_{\alpha\beta}}$, $\mathscr{L}_{,R_{\alpha\beta}} = \mathscr{L}_{,r_{\alpha\beta}}$, we see that the $H^{\alpha\beta}$ and $I^{\alpha\beta}$, introduced in this section as Lagrangean undetermined multipliers, are the same as the field variables $H^{\alpha\beta}$ and $I^{\alpha\beta}$ introduced in the previous discussion.

If

$$\det(\mathscr{L}_{,a_{\alpha\beta},a_{\mu\tau}}) \neq 0, \qquad \det(\mathscr{L}_{,r_{\alpha\beta},r_{\mu\tau}}) \neq 0, \qquad \det(\mathscr{L}_{,q_{(\varrho)\zeta},q_{(\varphi)\tau}}) \neq 0,$$

then equations (5.9) may be solved for $a_{\alpha\beta}$, $r_{\alpha\beta}$, and $q_{(\varrho)\zeta}$ as functions of $H^{\alpha\beta}$, $I^{\alpha\beta}$, $L^{\alpha}_{\beta\gamma}$, $W^{(\varrho)\zeta}$ and $Q_{(\varrho)}$. Using these solutions, we may eliminate $a_{\alpha\beta}$, $r_{\alpha\beta}$ and $q_{(\varrho)\zeta}$ in equation (5.8), thus obtaining

$$\mathscr{H} = \mathscr{H}(H^{\alpha\beta}, I^{\alpha\beta}, L^{\alpha}_{\beta\gamma}, W^{(\varrho)\zeta}, Q_{(\varrho)}, x). \tag{5.10}$$

Evaluating the variation of $\int\limits_{D^*} \mathscr{L}^* \, dV$ with respect to $H^{\alpha\beta}$, $I^{\alpha\beta}$, $L^{\alpha}_{\beta\gamma}$, $W^{(\varrho)\zeta}$, $Q_{(\varrho)}$, using the symmetry of $L^{\alpha}_{\beta\gamma}$, $I^{\alpha\beta}$ $\delta L^{\alpha}_{\beta\gamma}$ and the skewsymmetry of $H^{\alpha\beta}$, and equating the results to zero in accordance with equation (5.6) gives

$$0 = L^{\varepsilon}{}_{\varepsilon\beta,\alpha} - L^{\varepsilon}{}_{\varepsilon\alpha,\beta} - \mathscr{H}_{,H^{\alpha\beta}}, \tag{5.11}$$

$$0 = \tfrac{1}{2}(L^{\varepsilon}{}_{\varepsilon\alpha,\beta} + L^{\varepsilon}{}_{\varepsilon\beta,\alpha}) - L^{\varepsilon}{}_{\alpha\beta,\varepsilon} - \mathscr{H}_{,I^{\alpha\beta}}, \tag{5.12}$$

$$0 = Q_{(\varrho),\zeta} - \mathscr{H}_{,W^{(\varrho)\zeta}}, \tag{5.13}$$

$$0 = -W^{(\varrho)\zeta}{}_{,\zeta} - \mathscr{H}_{,Q_{(\varrho)}}, \tag{5.14}$$

$$0 = -2\mathscr{H}_{,L^{\zeta}_{\eta\lambda}} + 2\delta^{\lambda}_{\zeta} H^{\eta\varepsilon}{}_{,\varepsilon} + 2\delta^{\eta}_{\zeta} H^{\lambda\varepsilon}{}_{,\varepsilon} - \delta^{\lambda}_{\zeta} I^{\eta\varepsilon}{}_{,\varepsilon} - \delta^{\eta}_{\zeta} I^{\lambda\varepsilon}{}_{,\varepsilon} + 2I^{\eta\lambda}{}_{,\zeta}. \tag{5.15}$$

Equations (5.11) through (5.15) are the Hamiltonian form of the affine field equations.

Contracting equation (5.15) gives

$$0 = -2\mathscr{H}_{,L^{\varepsilon}_{\varepsilon\eta}} + 10 H^{\eta\varepsilon}{}_{,\varepsilon} - 3 I^{\eta\varepsilon}{}_{,\varepsilon}$$

from which we obtain

$$H^{\eta\varepsilon}{}_{,\varepsilon} = \tfrac{1}{10}(3 I^{\eta\varepsilon}{}_{,\varepsilon} + 2\mathscr{H}_{,L^{\varepsilon}_{\varepsilon\eta}}). \tag{5.16}$$

Thus, the Hamiltonian form of the exterior affine current is given by

$$S^{\eta} = \tfrac{1}{10}(3 I^{\eta\varepsilon}{}_{,\varepsilon} + 2\mathscr{H}_{,L^{\varepsilon}_{\varepsilon\eta}}). \tag{5.17}$$

Eliminating $H^{\eta\varepsilon}{}_{,\varepsilon}$ between equations (5.16) and (5.17) gives

$$I^{\zeta\eta}{}_{,\mu} - \tfrac{1}{5}\delta^{\eta}_{\mu} I^{\zeta\varepsilon}{}_{,\varepsilon} \cdot \tfrac{1}{5}\delta^{\zeta}_{\mu} I^{\eta\varepsilon}{}_{,\varepsilon} = \tfrac{1}{5}\delta^{\zeta}_{\mu} \mathscr{H}_{,L^{\varepsilon}_{\varepsilon\eta}} + \tfrac{1}{5}\delta^{\eta}_{\mu} \mathscr{H}_{.L^{\varepsilon}_{\varepsilon\zeta}} - \mathscr{H}_{,L^{\mu}_{\zeta\eta}}. \tag{5.18}$$

Equations (5.16) and (5.18) are equivalent to equations (5.15) and are considerably simpler to solve, since $\mathscr{H}_{,L^{\mu}_{\zeta\eta}}$ will be independent of $H^{\alpha\beta}$ in general.

6. Conservation Laws

The fundamental laws of physics are embodied in statements of the conservation of fundamental quantities: mass, momentum, energy, charge, *etc.* The mathematical formulation of these fundamental statements involves, however, certain integral relations of a topological rather than a metric differential geometric nature. In the theory of general relativity and also in EINSTEIN's unified field theory, conservation laws are embodied in the covariant conservation of a tensor $T^{\alpha\beta}$ of second rank, $T^{\alpha\beta}{}_{;\beta}=0$. To obtain the fundamental topological conservation laws from these theories, a nontensorial quantity $t^{\alpha}{}_{\beta}$ must be introduced by means of the identities satisfied by the tensorial fields, which, together with the mixed tensor density $\mathfrak{T}^{\alpha}{}_{\beta}$ corresponding to $T^{\alpha\beta}$, satisfies the equation

$$\int (\mathfrak{T}^{\alpha}{}_{\beta}+t^{\alpha}{}_{\beta})\, N_{\alpha}\, dS=0.$$

This is a most important example of the topological rather than the differential metric structure of mathematical statements of physics.

The present theory admits directly conservation laws of the required topological nature. It may be proved[14] that to any system of partial differential equations which arise from a variational statement constructed in a Hausdorff space, there corresponds a collection of functions $W^{\alpha}{}_{\beta}$ which satisfy the topological conservation laws $W^{\alpha}{}_{\beta,\alpha}=-\partial_{\beta}\mathscr{L}$. Specifically, set

$$W^{\alpha}{}_{\beta}=\mathscr{L}_{,L^{\zeta}_{\eta\lambda,\alpha}}L^{\zeta}{}_{\eta\lambda,\beta}+\mathscr{L}_{,Q_{(\varrho),\alpha}}Q_{(\varrho),\beta}-\delta^{\alpha}_{\beta}\mathscr{L}; \tag{6.1}$$

one may show that if the affine field equations are satisfied, then

$$W^{\alpha}{}_{\beta,\alpha}=-\partial_{\beta}\mathscr{L}. \tag{6.2}$$

Using equations (3.2) and (3.3) to evaluate the terms in equation (6.1), we have

$$W^{\alpha}{}_{\beta}=L^{\zeta}{}_{\zeta\mu,\beta}(2H^{\alpha\mu}+I^{\alpha\mu})+Q_{(\varrho),\beta}\mathscr{L}_{,Q_{(\varrho),\alpha}}-L^{\alpha}{}_{\zeta\mu,\beta}I^{\zeta\mu}-\delta^{\alpha}_{\beta}\mathscr{L}. \tag{6.3}$$

Equations (6.2) and (6.3) constitute the basic topological conservation laws of this affine field theory.

7. The Einstein Vacuum Fields as Exact Solutions of the Affine Field Equations in Empty Space

We now consider the affine field equations for the case of vacuum fields.

Definition. The space $\mathscr{E}$ will be said to be empty if and only if

$$S^{\alpha}=H^{\alpha\beta}=W^{(\varrho)\zeta}=Q_{(\varrho)}=A_{\alpha\beta}=0; \tag{7.1}$$

i.e., the Maxwellian field, the exterior affine current, and the affine matter fields vanish at all points in $\mathscr{E}$.

For an empty $\mathscr{E}$, we assume the following form for $\mathscr{L}$:

$$\zeta=\text{constant}, \qquad \det(r_{\alpha\beta})<0,$$

$$\mathscr{L}(0,r_{\alpha\beta},0,0,0,0)=2/\zeta\sqrt{-\det(r_{\alpha\beta})}. \tag{7.2}$$

Since equation (7.2) states that $\mathscr{L}$ is a scalar density, the resulting field equations will have tensorial nature, and the colon derivative becomes the semicolon derivative.

[14] EDELEN, D. G. B.: *op. cit.*, Section VII.

From equation (5.8) we have in this case

$$\mathscr{H} = I^{\alpha\beta}(L^{\mu}_{\ \mu\varepsilon} L^{\varepsilon}_{\ \alpha\beta} - L^{\mu}_{\ \varepsilon\alpha} L^{\varepsilon}_{\ \mu\beta} + r_{\alpha\beta}) - 2/\zeta \sqrt{-\det(r_{\alpha\beta})}. \tag{7.3}$$

Set

$$\det(r_{\alpha\beta}) = r^* \tag{7.4}$$

and

$$r^* r^{\alpha\beta} = r^*_{\ ,r_{\alpha\beta}}, \tag{7.5}$$

so that

$$\mathscr{L} = 2/\zeta \sqrt{-r^*}. \tag{7.6}$$

Substituting equation (7.6) into equation $(5.9)_2$ gives

$$I^{\alpha\beta} = 1/\zeta \sqrt{-r^*}\, r^{\alpha\beta}, \tag{7.7}$$

whence

$$\begin{aligned} I^{\alpha\beta} r_{\alpha\beta} &= 4/\zeta \sqrt{-r^*}, \\ I^{\alpha\beta} r_{\beta\gamma} &= 1/\zeta \sqrt{-r^*}\, \delta^{\alpha}_{\gamma}. \end{aligned} \tag{7.8}$$

Substituting (7.8) into (7.3), we have

$$\mathscr{H} = I^{\alpha\beta}(L^{\mu}_{\ \mu\varepsilon} L^{\varepsilon}_{\ \alpha\beta} - L^{\mu}_{\ \varepsilon\beta} L^{\varepsilon}_{\ \mu\alpha}) + 2/\zeta \sqrt{-r^*}. \tag{7.9}$$

The only Hamiltonian affine field equations not satisfied identically under equation (7.1) are

$$\tfrac{1}{2}(L^{\mu}_{\ \mu\alpha,\beta} + L^{\mu}_{\ \mu\beta,\alpha}) - L^{\mu}_{\ \alpha\beta,\mu} + L^{\mu}_{\ \varepsilon\beta} L^{\varepsilon}_{\ \mu\alpha} - L^{\varepsilon}_{\ \varepsilon\mu} L^{\mu}_{\ \alpha\beta} = 2/\zeta \left(\sqrt{-r^*}\right)_{,I^{\alpha\beta}} \tag{7.10}$$

and

$$2 I^{\alpha\beta}_{\ \ ;\gamma} - \delta^{\alpha}_{\gamma} I^{\beta\varepsilon}_{\ \ ;\varepsilon} - \delta^{\beta}_{\gamma} I^{\alpha\varepsilon}_{\ \ ;\varepsilon} = 0. \tag{7.11}$$

Contracting equation (7.11) with respect to (β, γ) gives

$$I^{\alpha\gamma}_{\ \ ;\gamma} = 0$$

so that equation (7.11) is equivalent to

$$I^{\alpha\beta}_{\ \ ;\gamma} = 0. \tag{7.12}$$

Differentiating equation $(7.8)_1$ with respect to $I^{\alpha\beta}$, we have

$$2 r_{\alpha\beta} = 4/\zeta \left(\sqrt{-r^*}\right)_{,I^{\alpha\beta}}. \tag{7.13}$$

Substituting this result into equation (7.10) and using the definition of $R_{\alpha\beta}$ gives

$$R_{\alpha\beta} = r_{\alpha\beta}, \tag{7.14}$$

in agreement with equation (5.4), as was to be expected. Substituting equation (7.7) into equation (7.12) gives

$$\left(\sqrt{-r^*}\, r^{\alpha\beta}\right)_{;\gamma} = 0. \tag{7.15}$$

Equations (7.14) and (7.15) are the affine field equations for empty space.

Expanding equation (7.15), we have

$$\sqrt{-r^*}\, r^{\alpha\beta}_{\ \ ;\gamma} + r^{\alpha\beta} \left(\sqrt{-r^*}\right)_{;\gamma} = 0$$

or which it is sufficient to require

$$\sqrt{-r^*}\, r_{\alpha\beta;\gamma} + r_{\alpha\beta} \left(\sqrt{-r^*}\right)_{;\gamma} = 0, \tag{7.16}$$

upon noting that $r^{\alpha\beta} r_{\beta\gamma} = \delta^\alpha_\gamma$. Taking determinants of $\sqrt{-r^*}\, r_{\alpha\beta} = M_{\alpha\beta}$, we see that equation (7.16) implies

$$(\sqrt{-r^*})_{;\gamma} = 0. \tag{7.17}$$

Since $A_{\alpha\beta} = 0$ by hypothesis, equation (7.17) is admissible in that, by Theorem 1, a covariant constant scalar density field can exist in $\mathscr{E}$. Thus equation (7.16) yields

$$r_{\alpha\beta;\gamma} = 0. \tag{7.18}$$

Since $r_{[\alpha\beta]} = 0$, and since $\det(r_{\alpha\beta})$ has been assumed to be non-zero, $r_{\alpha\beta}$ satisfies equations (2.8) through (2.10), and hence $\mathscr{E}$ reduces to a metric space under (7.17) and (7.18). Thus, by equation (2.15) we have

$$L^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} \tag{7.19}$$

since by (7.18) $Q_{\alpha\beta\gamma} = 0$. In this case one immediately obtains

$$L^\varepsilon{}_{\varepsilon\lambda} = (\ln \sqrt{-r^*})_{,\lambda}$$

so that (7.18) implies (7.17). To this point the functions $r_{\alpha\beta}$ have been arbitrary smooth functions. Under equation (7.19), we have

$$R_{\alpha\beta} = J_{\alpha\rho} \tag{7.20}$$

by equation (2.19). Substituting this result into equation (7.14), we have

$$J_{\alpha\beta} = r_{\alpha\beta}, \tag{7.21}$$

which is a system of ten partial differential equations of the second order for determination of the ten potential functions $r_{\alpha\beta}$.

Theorem 7. *A general class of solutions to the affine field equations in an empty space with Lagrangean function given by equation* (7.2) *is*

$$L^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} \tag{7.22}$$

where (1) *the* $\Gamma^\alpha{}_{\beta\gamma}$ *are Christoffel symbols based on the tensor* $r_{\alpha\beta}$, (2) *the* $r_{\alpha\beta}$ *are any solution to the equations*

$$\begin{aligned} J_{\alpha\beta} &= r_{\alpha\beta}, \\ r_{[\alpha\beta]} &= 0, \\ \det(r_{\alpha\beta}) &\neq 0, \end{aligned} \tag{7.23}$$

and (3) $J_{\alpha\beta}$ *is the third contraction of the affine curvature tensor associated with the connection* $\Gamma^\alpha{}_{\beta\gamma}$.

By equation (2.16) $\Gamma^\alpha{}_{\beta\gamma}$ is homogeneous of degree zero in the $r_{\alpha\beta}$ and their derivatives. Thus, by equations (2.5) and (2.19) $J_{\alpha\beta}$ is homogeneous of degree zero in $r_{\alpha\beta}$ and its derivatives, so that equation (7.23) is equivalent to

$$\begin{aligned} J_{\alpha\beta} &= \Lambda' r_{\alpha\beta}, \\ \Lambda' r_{\alpha\beta} &= r_{\alpha\beta} \end{aligned} \tag{7.24}$$

for arbitrary $\Lambda \neq 0$ [15]. Equations (7.24) are just the vacuum field equations of EINSTEIN'S general theory of relativity with cosmological constant. Thus any solution to EINSTEIN'S vacuum equations with cosmological constant Λ gives an $'r_{\alpha\beta}$ which when substituted into equation (7.22) constitutes an exact solution of the vacuum affine field equations.

It is of interest to note that equation (7.24) exhibits the following non-singular solution [16]:

$$r_{11} = r_{22} = r_{33} = -e^{-2x^4\sqrt{\Lambda/3}}; \qquad r_{44} = c^2. \tag{7.25}$$

Thus the space resulting from the affine field equations under equation (7.1) is approximately Minkowskian when Λ is sufficiently small and x^4 is restricted to a small enough interval.

If we require the affine field equations to be satisfied everywhere except at the origin, then we obtain the solution [17]

$$r_{11} = r_{22} = r_{33} = -c^2/r_{44} = -1/(1 - 2m/r - \Lambda r^2/3)$$
$$r = x^{1^2} + x^{2^2} + x^{3^2}$$

which, of course, represents empty space except in the vicinity of the origin. In this case, however, we obtain the three classical tests of relativity to within the same violations of the field equations (*i.e.*, at the origin) as in EINSTEIN'S general theory of relativity.

8. Semi-Empty Space and the Fields Resulting from Scalar Density Lagrangean Functions

We now consider the affine field equations for the case of semi-empty space.

Definition. The space $\mathscr{E}$ will be said to be **semi-empty** if and only if

$$S^\alpha = Q_{(\varrho)} = W^{(\varrho)\beta} = 0; \tag{8.1}$$

i.e., the exterior affine current and the affine matter fields vanish at all points of $\mathscr{E}$.

Comparing the definitions of empty space and semi-empty space, we see that a semi-empty space reduces to an empty space if $A_{\alpha\beta} = 0$.

Definition. A semi-empty space $\mathscr{E}$ will be said to be proper if and only if $A_{\alpha\beta} \neq 0$ for some (α, β). We confine our attention to a proper semi-empty space.

In the last section, we based our considerations on a particular Lagrangean function, partly for expediency and partly because by the very nature of empty space the scope of the problem was narrowed. In this section, we adopt the alternate course of analyzing a general class of Lagrangean functions.

Consider the class of Lagrangean functions which are scalar densities and which are such that

$$\partial_{L^\alpha_{\beta\gamma}} \mathscr{L} = 0. \tag{8.2}$$

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[15] That Λ must be non-zero is evident, since for $\Lambda = 0$ the Hamiltonian form of the field equations is invalid, *i.e.*, $\det(\mathscr{L}_{,R_{\alpha\beta}, R_{\gamma\delta}}) = 0$.

[16] MCVITTIE, G. C.: General Relativity and Cosmology, pp. 73. London: Chapman & Hall Ltd. 1956.

[17] TOLMAN, R. C.: Relativity Thermodynamics and Cosmology, p. 345. Oxford: At the Clarendon Press 1934.

When $\mathscr{L}$ is a member of this class we see from equations (3.2) and (3.3) that

$$I^{\alpha\beta} = \mathscr{L}_{,R_{\alpha\beta}} \tag{8.3}$$

and

$$H^{\alpha\beta} = \mathscr{L}_{,A_{\alpha\beta}} \tag{8.4}$$

are contravariant tensor densities. The affine field equations are thus tensorial, and $I^{\alpha\beta}{}_{:\gamma} \equiv I^{\alpha\beta}{}_{;\gamma}$. By equations (4.2) and (8.2) we can satisfy the condition $S^\alpha = 0$ only if

$$I^{\alpha\varepsilon}{}_{;\varepsilon} = 0. \tag{8.5}$$

Substituting equations (8.1) and (8.5) into the affine field equations (3.7) and (3.8), we have

$$H^{\alpha\varepsilon}{}_{,\varepsilon} = 0, \tag{8.6}$$

$$I^{\alpha\beta}{}_{;\gamma} = 0 \tag{8.7}$$

as the affine field equations in semi-empty space governed by scalar density Lagrangean functions. Equations (8.6) are just the second set of the Maxwell equations in the absence of exterior affine current. Equations (8.7) are of a different nature and will be seen to determine systems of potential-like functions.

In order that $\mathscr{E}$ be a *proper* semi-empty space, we must have

$$A_{\alpha\beta} \neq 0 \tag{8.8}$$

for at least one choice of (α, β). Thus, by equation (4.5) there exists a non-zero affine vector potential $f_\gamma \neq \varphi_{,\gamma}$ such that

$$f_{[\alpha,\beta]} = A_{\alpha\beta}. \tag{8.9}$$

Expanding the right side of equation (8.9) by means of equation (3.6), we have

$$f_{[\alpha,\beta]} = 2L^{\varepsilon}{}_{\varepsilon[\beta,\alpha]}. \tag{8.10}$$

Solving equation (8.10) for $L^{\varepsilon}{}_{\varepsilon\beta}$, we see that

$$L^{\varepsilon}{}_{\varepsilon\beta} = -\tfrac{1}{2} f_\beta + \psi_{,\beta} \tag{8.11}$$

is a necessary condition for $\mathscr{E}$ to be a proper semi-empty space.

There are basically two ways in which equation (8.11) can be imbedded in a general representation for the components of affine connection:

1)
$$L^{\alpha}{}_{\beta\gamma} = Z^{\alpha}{}_{\beta\gamma} - \tfrac{1}{10}\left(f_\beta\, \delta^\alpha_\gamma + f_\gamma\, \delta^\alpha_\beta\right) \tag{8.12}$$

where $Z^{\alpha}{}_{\beta\gamma}$ is an arbitrary affinity constrained by the conditions

$$Z^{\alpha}{}_{\alpha\gamma} = \psi_{,\gamma}; \tag{8.13}$$

2)
$$L^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + a\left(f_\beta\, \delta^\alpha_\gamma + f_\gamma\, \delta^\alpha_\beta\right) + b\, f_\mu\, g^{\mu\alpha}\, g_{\beta\gamma}, \tag{8.14}$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ are Christoffel symbols based on the symmetric tensor $g_{\alpha\beta}$ whose determinant does not vanish and

$$5a + b = -\tfrac{1}{2}. \tag{8.15}$$

These alternate forms of the affinity are those which are required for representation of the two classes of solutions to the affine field equations in semi-empty space.

If $I^{\alpha\beta} \neq 0$, (8.6) and (8.7) are 44 equations of which only 40 are independent (*cf.* Theorem 4). Thus, if the affinity (8.12) is used, the 40 independent field equations together with the 4 conditions (8.13) yield 44 equations for the determination of the 44 functions $Z^{\alpha}_{\beta\gamma}$ and f_γ. It is evident that for $I^{\alpha\beta} \neq 0$ the use of (8.14) would yield an overdetermined system since there would be only 14 functions $g_{\alpha\beta}$ and f_γ to be determined by the 44 field equations. We have thus proved

Theorem 8. *The first class of solutions to the affine field equations in a proper semi-empty space with a scalar density Lagrangean function satisfying equation* (8.2) *is given by*

$$L^{\alpha}_{\beta\gamma} = Z^{\alpha}_{\beta\gamma} - \tfrac{1}{10}\,(f_\beta\, \delta^{\alpha}_{\gamma} + f_\gamma\, \delta^{\alpha}_{\beta})$$

where the 44 functions $Z^{\alpha}_{\beta\gamma}$ and f_γ are any solutions to the equations

$$(\mathscr{L}_{,A_{\alpha\varepsilon}})_{,\varepsilon} = 0,$$

$$(\mathscr{L}_{,R_{\alpha\beta}})_{;\gamma} = 0, \qquad \mathscr{L}_{,R_{\alpha\beta}} \neq 0,$$

$$Z^{\alpha}_{\alpha\gamma} = \psi_{,\gamma}.$$

This class of solutions is principally of interest when we consider interacting electromagnetic fields and is the only possible class of solutions when the exterior affine current is a non-zero vector.

The second class of solutions to equations (8.6) and (8.7) is obtained from the fact $I^{\alpha\beta} = 0$ is an exact formal solution of equation (8.7). If this formal solution is algebraically consistent, equations (8.6) and (8.7) reduce to the 14 equations

$$H^{\alpha\varepsilon}{}_{,\varepsilon} = 0, \tag{8.16}$$

$$I^{\alpha\beta} = 0. \tag{8.17}$$

Using the affinity (8.14), we obtain a deterministic system since equations (8.16) and (8.17) are 14 equations for the determination of the 14 unknown functions $g_{\alpha\beta}$ and f_γ. On the other hand, if (8.12) is used, we have only 14 field equations (8.16) and (8.17) plus the four equations (8.13) for the determination of the 44 unknown functions $Z^{\alpha}_{\beta\gamma}$ and f_γ. The affine field equations would then only determine the affinity to within 30 arbitrary functions of position. It is thus evident that the affinity corresponding to (8.16) and (8.17) must be given by equation (8.14) if a deterministic system is to result.

Since a and b in equation (8.14) are arbitrary to within equation (8.15), we may choose them so that equations (8.14) and (2.15) yield a unique tensor $Q_{\gamma\alpha\beta}$. Set $b = -a$ so that a has the value $-\frac{1}{8}$ by equation (8.15). For this choice, equation (8.14) becomes

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \tfrac{1}{8}\,(f_\beta\, \delta^{\alpha}_{\gamma} + f_\gamma\, \delta^{\alpha}_{\beta} - f_\mu\, g^{\mu\alpha}\, g_{\beta\gamma}). \tag{8.18}$$

Comparing equations (2.15) and (8.18), it is evident that

$$Q_{\gamma\alpha\beta} = -\tfrac{1}{4}\, f_\gamma\, g_{\alpha\beta}.$$

From equation (2.13) we thus obtain the following evaluation of the covariant derivative of $g_{\alpha\beta}$:

$$g_{\alpha\beta;\gamma} = \tfrac{1}{4} f_\gamma g_{\alpha\beta} = -Q_{\gamma\alpha\beta}, \tag{8.19}$$

and hence the field tensor $R_{\alpha\beta}$ is expressed in terms of f_γ, $g_{\alpha\beta}$, and their derivatives by equation (2.19). The affine connection determined by equation (8.19), namely (8.18), is that of WEYL's geometry[18] where $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is the fundamental quadratic form and $-\frac{1}{4} f_\gamma\, dx^\gamma$ is the fundamental linear form. If $g_{\alpha\beta}$ undergoes a conformal transformation

$$'g_{\alpha\beta} = \sigma(x)\, g_{\alpha\beta},$$

then

$$'g_{\alpha\beta;\gamma} = \left(\tfrac{1}{4} f_\gamma \sigma + \sigma_{,\gamma}\right) g_{\alpha\beta} = \left(\tfrac{1}{4} f_\gamma + (\ln\sigma)_{,\gamma}\right) 'g_{\alpha\beta}.$$

Hence, if we use $'g_{\alpha\beta}$ in computing $\Gamma^\alpha{}_{\beta\gamma}$ instead of $g_{\alpha\beta}$ and if at the same time f_γ is transformed into

$$'f_\gamma = f_\gamma + 4\,(\ln\sigma)_{,\gamma}, \tag{8.20}$$

we get the same affinity (8.18). This implies that the tensor $g_{\alpha\beta}$ is fixed only to within an arbitrary gauge factor $\sigma(x)$ and that the affine vector potential f_γ is determined only to within the transformations (8.20). This arbitrariness in the choice of gauge is just what should be expected, since the Maxwellian field $A_{\alpha\beta}$ must be gauge invariant if it is to correspond to the field tensor of classical electromagnetism. We have thus proved

Theorem 9. *The second class of solutions to the affine field equations in a proper semi-empty space with a scalar density Lagrangean function satisfying equation* (8.2) *is given by*

$$L^\alpha_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \tfrac{1}{8}\left(f_\gamma\, \delta^\alpha_\beta + f_\beta\, \delta^\alpha_\gamma - f_\mu\, g^{\mu\alpha} g_{\beta\gamma}\right), \tag{8.21}$$

where (1) f_γ *are the components of the affine vector potential,* (2) $\Gamma^\alpha{}_{\beta\gamma}$ *are Christoffel symbols based on the symmetric tensor* $g_{\alpha\beta}$ *whose determinant does not vanish, and* (3) f_γ *and* $g_{\alpha\beta}$ *are any solutions to the potential equations*

$$H^{\alpha\varepsilon}{}_{,\varepsilon} = 0, \tag{8.22}$$

$$I^{\alpha\beta} = 0 \tag{8.23}$$

under which $g_{\alpha\beta}$ *is determined to within an arbitrary gauge factor* $\sigma(x)$ *and the affine vector potential is determined to within the gauge transformation*

$$'f_\gamma = f_\gamma + 4\,(\ln\sigma)_{,\gamma}. \tag{8.24}$$

9. Classic Vacuum Electrodynamics as Exact Solutions of the Affine Field Equations

We assume $\mathscr{E}$ to be a proper semi-empty space and set

$$\mathscr{L} = \tfrac{1}{2}\sqrt{-h}\, A_{\alpha\beta} A_{\delta\gamma}\, h^{\alpha\delta} h^{\beta\gamma} \tag{9.1}$$

[18] WEYL, H.: Math. Zeitschr. 2, 384—411 (1918). SCHOUTEN: *op. cit.*, p. 133.

where $h_{\alpha\beta}$ is a symmetric tensor of signature 2,

$$h = \det(h_{\alpha\beta}) < 0, \tag{9.2}$$

and

$$h^{\alpha\beta} h = h_{,h_{\alpha\beta}}. \tag{9.3}$$

From equations (3.2) and (3.3) we have

$$H^{\alpha\beta} = \sqrt{-h}\, A_{\delta\gamma} h^{\alpha\delta} h^{\beta\gamma} \tag{9.4}$$

and

$$I^{\alpha\beta} \equiv 0. \tag{9.5}$$

Since the Lagrangean given by (9.1) is a scalar density which satisfies equation (8.2) and since, by equation (9.5), $I^{\alpha\beta} = 0$ cannot yield algebraic inconsistency, Theorem 9 is applicable. Thus

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \tfrac{1}{8}\left(f_\gamma \delta^\alpha_\beta + f_\beta \delta^\alpha_\gamma - f_\mu g^{\mu\alpha} g_{\beta\gamma}\right),$$

and the functions $g_{\alpha\beta}$ and f_γ are to be determined by equations (9.4) and (9.5). The identical vanishing of $I^{\alpha\beta}$ places no constraint on the choice of the functions $g_{\alpha\beta}$, and hence we may set $g_{\alpha\beta} = h_{\alpha\beta}$ with no loss of generality. Thus, since the tensor $h_{\alpha\beta}$ can be chosen at will and the tensor $g_{\alpha\beta}$ has been shown to be the coefficient tensor of the fundamental quadratic form, there is no restriction on the choice of coordinate system for the solution of the remaining field equations

$$H^{\alpha\varepsilon}{}_{,\varepsilon} = 0. \tag{9.6}$$

Equations (9.4) and (9.6) together with $A_{\alpha\beta} = f_{[\alpha,\beta]}$ are just the field equations for the classic vacuum electromagnetic field. Moreover, the Lagrangean function (9.1) together with the field tensors $A_{\alpha\beta}$ and $H^{\alpha\beta}$ are invariant under all gauge transformations of class C^1; the Lorentz condition can be used to fix the gauge by solving

$$'f^{\gamma}{}_{;\gamma} = 0 \tag{9.7}$$

for $(\ln\sigma)$ and substituting into equation (8.24); and $\mathscr{L}_{,h_{\alpha\beta}}$ yields the field momentum energy tensor

$$A_{\lambda\eta} A_{\varrho\tau} h^{\lambda\varrho} \left(h^{\eta\tau} h^{\alpha\beta}/4 - h^{\eta\alpha} h^{\tau\beta}\right).$$

We have thus proved

Theorem 10. *The solution manifold of the affine field equations which results from the Lagrangean function* (9.1) *in a proper semi-empty space is isomorphic with the solution manifold of classic vacuum electrodynamics.*

Theorem 10 combined with the results of Section 7 shows that for one choice of the Lagrangean function the affine field equations describe the classic vacuum electromagnetic field, and for another choice they describe the classic vacuum gravitational field. The Lagrangean function thus plays the role of a constitutive function for the gravitational field as well as the electromagnetic field. This result is analogous to the situation in which the continuum equilibrium equations may describe either a solid or a fluid relative to the choice of a constitutive function. We thus see that *the classic vacuum gravitational and electromagnetic*

fields are both described by the affine field equations and correspond to a simple separation of effects through the choice of the constitutive (Lagrangean) function for the particular field under investigation.

The general problem of interacting electromagnetic and gravitational fields could be treated here on a formalistic basis through use of Section 8 and a study of various possible Lagrangean functions. An obvious first choice for $\mathscr{L}$ would be a linear combination of (7.2) and (9.1), which may be readily solved by the use of Theorem 9. However, this choice together with the more general forms of the Lagrangean function does not offer a clear cut decision as to which yields a more adequate picture of the true fields. It is felt that this lack of definiteness arises basically from the problem of a consistent and rational description of elementary matter. Since this problem exceeds the scope of this paper, we leave the examination of interacting electromagnetic and gravitational fields to a future communication.

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Beiträge zu einer nichtlinearen Theorie der Stabilität von Schichtenströmungen längs zylindrisch gekrümmter Wände gegenüber dreidimensionalen Störungen

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Einleitung

Die vollständige Erforschung der Strömungsvorgänge beim laminar-turbulenten Umschlag ist ohne das Studium endlicher Störungen nicht denkbar. Abgesehen von den Störungen, deren Amplituden schon im Zeitpunkt ihrer Entstehung groß gegen die Grundströmung oder vergleichbar mit ihr sind und daher ohne Berücksichtigung der vollen nichtlinearen Grundgleichungen überhaupt nicht behandelt werden können, gelten auch die Aussagen einer für kleine Störungen entwickelten und bisher fast ausschließlich angewandten linearisierten Theorie im allgemeinen nur für kleine Zeitintervalle. Versucht man nämlich, die Aussagen dieser Theorie auf beliebige Zeitspannen auszudehnen, so gerät man, wie in dieser Arbeit gezeigt werden soll, in Widerspruch zu den Ergebnissen einer allgemeinen Behandlung der nichtlinearen Stabilitätsphänomene.

Erste Aussagen einer strengen nichtlinearen Stabilitätstheorie bewies J. SERRIN* in [*1*], wo in aller Strenge nachgewiesen wird, daß eine beliebige Strömung in einem beschränkten Raum gegenüber allen denkbaren Störungen sicher dann stabil ist, wenn die mit dem Maximalwert der Grundströmungsgeschwindigkeit und mit dem Durchmesser des räumlichen Bereichs gebildete Reynoldszahl kleiner als die Zahl 5,71 ist. (Hinreichendes Stabilitätskriterium.) Grundströmung und Störungen dürfen dabei von allen drei Ortskoordinaten und der Zeit abhängen. Die einzigen Voraussetzungen bilden gewisse, mit den hydrodynamischen Grundgleichungen stets verbundene Differenzierbarkeitseigenschaften.

Als wichtigen Anwendungsfall gab SERRIN für die Strömung zwischen zwei rotierenden koaxialen Zylindern eine im Hinblick auf die Allgemeinheit der zugelassenen Störungsklasse erstaunlich gute hinreichende Stabilitätsgrenze an. Wir kommen darauf im letzten Paragraphen eingehender zurück.

Eine Erweiterung des Gesichtskreises der bisherigen Stabilitätstheorie durch eine nichtlineare Behandlung der aufgeworfenen Probleme geben auch die in neuerer Zeit erschienenen theoretischen Untersuchungen von D. MEKSYN und

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* Auf die zitierte Arbeit von J. SERRIN wurde ich dankenswerterweise von C. TRUESDELL hingewiesen.

J. T. Stuart. Sie gewinnen, nachdem sie gewisse, in ihren Konsequenzen schwer übersehbare Annahmen über Gestalt und zeitliches Verhalten der Störungen gemacht haben, durch Berechnung des nichtperiodischen Anteils der gestörten Grundströmung gute Übereinstimmung ihrer Ergebnisse mit experimentellen Resultaten von G. I. Taylor (vgl. hierzu J. T. Stuart [*2*]).

Nach den Aussagen der Stabilitätstheorie kleiner Störungen, wie sie von G. I. Taylor [*3*], W. Tollmien [*4*], H. Görtler [*5*] u. a. entwickelt wurde (vgl. hierzu auch den zusammenfassenden Bericht von C. C. Lin [*6*]), müßten die Störungen im überkritischen Bereich, also beim Überschreiten eines kritischen Parameters (meist die Reynoldszahl) exponentiell mit der Zeit über alle Schranken wachsen. Die in den genannten Arbeiten verwendeten Störungsansätze beschreiben jedoch das zeitliche Verhalten angefachter Störungen nur für hinreichend kleine Zeiten richtig, wo die Störungen noch klein gegen die Grundströmung sind.

Ausgehend von einer auch für endliche Störungen sinnvollen Verallgemeinerung der Taylor-Görtler-Wirbel werden wir, unter Berücksichtigung der vollen Navier-Stokesschen Gleichungen, obere und untere Schranken für sämtliche Störungskomponenten angeben. Diese Schranken, die eine Grenze für das zeitliche Anwachsen der Störungen darstellen, sind endlich, falls die Grundströmung selbst, die auch *instationär* sein darf, für alle Zeiten beschränkt bleibt.

Im letzten Paragraphen werden die hier angewandten Abschätzungsmethoden zur Herleitung eines hinreichenden Stabilitätskriteriums benutzt. Das Ergebnis, welches allerdings nur für Störungen des hier betrachteten Typs gilt, wird in einem Spezialfall mit einem Resultat von J. Serrin [*1*] verglichen.

Wir setzen in dieser Arbeit, neben einigen physikalisch leicht vertretbaren Stetigkeits- und Differenzierbarkeitseigenschaften, lediglich voraus, daß sowohl die Grundströmung wie auch die Störungskomponenten von der Hauptströmungsrichtung unabhängig sind, wodurch insbesondere Störungen vom Tollmien-Schlichtingschen Typ ausgeschlossen werden.

Die Untersuchungen stützen sich wesentlich auf den Nagumo-Westphalschen Satz für nichtlineare parabolische Differentialgleichungen, der von H. Görtler [*7*] und K. Nickel [*8*] in die Grenzschichttheorie eingeführt wurde. Im Anhang zu dieser Arbeit sind die Aussage dieses Satzes sowie einige Bezeichnungen, die sich bei seiner Anwendung eingebürgert haben und hier verwendet werden, zusammengestellt.

§ 1. Allgemeine Vorbemerkungen

Wir betrachten im folgenden stets Strömungen längs einer zylindrisch gekrümmten Fläche S, die den konstanten Krümmungsradius R besitze. Zweckmäßigerweise benutzen wir das in Abb. 1 dargestellte orthogonale Dreibein, dessen Ursprung in einen nicht genauer zu fixierenden Punkt von S gelegt wird. Die x-Achse zeigt dabei in Hauptströmungsrichtung und die y-Achse zum Krümmungsmittelpunkt hin, während die z-Richtung mit einer Zylindererzeugenden zusammenfällt. u, v, w bezeichnen die Geschwindigkeiten in x-, y-, z-Richtung.

Unsere Betrachtungen beschränken sich stets auf den Bereich $y \geqq 0$ und beinhalten daher zunächst nur Aussagen über Strömungen längs konkav gekrümmter Wände. Die Erweiterung auf konvex gekrümmte Wände ist leicht durchzuführen, interessiert jedoch in diesem Zusammenhang nicht.

Ferner gelten unsere Überlegungen auch im Grenzfall $R=\infty$, also für die längs überströmte ebene Platte, ohne daß wir dies jedesmal gesondert hervorheben werden.

Wir wollen unsere Untersuchungen auf solche Grundströmungen u_0 beschränken, die Funktionen von y und t allein sind, also von x und z nicht abhängen.

$$u_0 = u_0(y,t), \qquad v_0 = w_0 = 0. \tag{1.1}$$

Für die Druckverteilung ergibt sich dann

$$p_0 = -\int \frac{u_0^2(y,t)}{R-y}\, dy. \tag{1.1a}$$

Hierunter fallen nun insbesondere alle stationären und instationären Strömungen in der Taylorschen Versuchsanordnung, bevor die bekannte Wirbelinstabilität einsetzt. Um dies einzusehen, identifiziere man einfach den äußeren Zylinder mit der in Abb. 1 gezeichneten Fläche S.

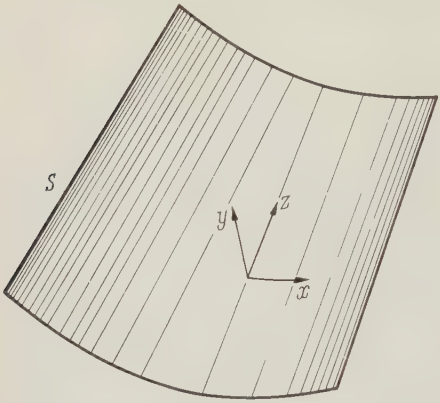

Abb. 1. Das benutzte Koordinatensystem

Von der Geschwindigkeitsverteilung, die eine Lösung der Navier-Stokesschen Gleichungen ist, müssen wir fordern, daß sie im Innern des Strömungsgebietes zweimal nach y und einmal nach t differenzierbar ist. Die Ableitungen seien im abgeschlossenen Strömungsgebiet stetig. (Die Begriffe der Stetigkeit und Differenzierbarkeit in den stets zugrunde liegenden unbeschränkten Gebieten werden unten erklärt.)

Es erweist sich nun im folgenden als zweckmäßig, Grundgleichungen, Anfangs- und Randbedingungen in dimensionsloser Form zu schreiben. Daher machen wir von vornherein y und z mit einer für jedes Problem charakteristischen Länge h, die Geschwindigkeitskomponenten u, v, w mit einer problemabhängigen Bezugsgeschwindigkeit U_0 und den Druck p mit ϱU_0^2 ($\varrho =$ Dichte) dimensionslos. Da im folgenden alle genannten Größen nur in dimensionsloser Gestalt verwendet werden, erübrigt sich die Einführung neuer Bezeichnungen; lediglich für den dimensionslosen Krümmungsradius setzen wir: $\overline{R}=R/h$.

Nachdem wir angenommen haben, die Grundströmung sei unabhängig von der x- und z-Koordinate, müssen hier nur noch zwei Strömungsgebiete unterschieden werden: das von zwei in z-Richtung unendlich langen, koaxialen Kreiszylindern begrenzte Gebiet $G_1=\{0<y<a;\ -\infty<z<+\infty\}$, ($a=$ konst.) und das von der halbunendlichen ebenen Platte begrenzte Gebiet $G_2=\{0<y<\infty;\ -\infty<z<+\infty\}$. Erwähnt sei noch, im Hinblick auf die Anwendung des Nagumo-Westphalschen Satzes, daß beide Gebiete sich topologisch auf das beschränkte Gebiet $G=\{0<y<1;\ -1<z<1\}$ abbilden lassen.

Führt man jetzt noch senkrecht zur y, z-Ebene die Zeitkoordinate t ein (mit h/U_0 dimensionslos gemacht), dann bilden die Punkte (y, z, t) mit den Eigen-

schaften: $(y, z) \in G_i$ und $t \in I = \{0 < t < T\}$ das Innere eines Zylinders Z_i $(i = 1, 2)$. Seine Mantelfläche bezeichnen wir mit S_i und seine Grundfläche (Schnitt mit der Ebene $t = 0$) mit G_i^*; S_i und G_i^* enthalten dabei unendlich ferne Punkte. Daher müßten die Begriffe: Randwert, Stetigkeit und Differenzierbarkeit einer Funktion in den hier zugrunde liegenden unbeschränkten Gebieten genau definiert werden. Dies soll im Anhang dieser Arbeit geschehen, und zwar in der von K. Nickel in [8] angegebenen Weise, mit der ihm die Übertragung des Nagumo-Westphalschen Satzes auf nichtbeschränkte Gebiete gelang. Als wesentlich erweist sich dabei nur eine Voraussetzung, daß sich nämlich das unbeschränkte Gebiet topologisch auf ein beschränktes Gebiet abbilden läßt.

Nachdem dies aber für die hier betrachteten Gebiete gesichert ist, wie wir oben schon erwähnt haben, können wir die oben eingeführten und im folgenden benutzten Bezeichnungen in Strenge rechtfertigen.

Der Grundströmung u_0 überlagern wir nun Störungen mit den Komponenten u_1, v_1 und w_1, die wiederum von der Hauptströmungsrichtung unabhängig sind. Wir setzen also:

$$\begin{aligned} u(y,z,t) &= u_0(y,t) + u_1(y,z,t) \\ v(y,z,t) &= \phantom{u_0(y,t) +{}} v_1(y,z,t) \\ w(y,z,t) &= \phantom{u_0(y,t) +{}} w_1(y,z,t) \\ p(y,z,t) &= p_0(y,t) + p_1(y,z,t)\,. \end{aligned} \tag{1.1}$$

Die Differentialgleichungen für die Störungskomponenten, die man aus (1.1) und den Navier-Stokesschen Gleichungen erhält, wenn man berücksichtigt, daß u_0 und p_0 diese Gleichungen befriedigen, werden erst in den folgenden Paragraphen angegeben, wo die Komponenten getrennt abgeschätzt werden.

Die Anfangs- und Randbedingungen lauten, wenn $u_1^{(0)}$, $v_1^{(0)}$ und $w_1^{(0)}$ die Anfangsverteilungen der Störungskomponenten bezeichnen:

$$\begin{aligned} & u_1(y,z,t) = u_1^{(0)}(y,z)\,; \\ & v_1(y,z,t) = v_1^{(0)}(y,z)\,; \\ & w_1(y,z,t) = w_1^{(0)}(y,z)\,, \quad \text{wenn} \quad (y,z,t) \in G_i^*, \qquad (i=1,2)\,; \\ & u_1(y,z,t) = v_1(y,z,t) = w_1(y,z,t) = 0\,, \quad \text{wenn} \quad (y,z,t) \in S_i \qquad (i=1,2)^{\star} \end{aligned} \tag{1.2}$$

ist. Ferner sollen die in den Differentialgleichungen auftretenden Ableitungen in Z_i existieren und im abgeschlossenen Bereich $\bar{Z}_i$, der aus Z_i durch Hinzunahme sämtlicher Randpunkte entsteht, stetig sein. Die in §3 und §4 benutzten uneigentlichen Integrale über $\bar{G}_i$, wo $\bar{G}_i$ das durch seine Randpunkte abgeschlossene Gebiet G_i ist, sollen gleichmäßig konvergieren, so daß eine Vertauschung der Integrationsreihenfolge stets erlaubt ist.

⋆ Die folgenden Untersuchungen behalten jedoch ihre Gültigkeit, wenn man statt des physikalisch sinnvollen und in (1.2) geforderten Verschwindens der Störungen für $z \to \pm\infty$, Periodizität in z-Richtung verlangt, wie sie bei den linearisierten Überlegungen stets auftrat.

§ 2. Obere und untere Schranken für das zeitliche Anwachsen der ersten Störungskomponente u_1

In diesem Paragraphen beschäftigen wir uns ausschließlich mit der ersten Störungskomponente u_1 und schätzen sie mit Hilfe des Nagumo-Westphalschen Satzes in einer außerordentlich einfachen Weise nach oben und unten ab.

Die Differentialgleichung für die erste Komponente u_1 lautet (vgl. H. GÖRTLER [5]):

$$\frac{\partial u_1}{\partial t} = \frac{1}{Re} L[u_1] + \left(-\frac{\partial u_0}{\partial y} + \frac{u_0}{\bar{R}-y} - \frac{\partial u_1}{\partial y} + \frac{u_1}{\bar{R}-y}\right) v_1 - w_1 \frac{\partial u_1}{\partial z}, \tag{2.1}$$

mit dem elliptischen Operator:

$$L \equiv \frac{\partial^2}{\partial y^2} - \frac{1}{\bar{R}-y}\frac{\partial}{\partial y} - \frac{1}{(\bar{R}-y)^2} + \frac{\partial^2}{\partial z^2}$$

und mit $Re = \frac{U_0 h}{\nu}$. ($\nu =$ kinematische Zähigkeit.) Die Anfangs- und Randbedingungen wurden in den Gleichungen (1.2) angegeben.

Wir konstruieren nun Ober- und Unterfunktionen* $\hat{u}$ bzw. $\check{u}$, die unabhängig von der Gestalt der Störungskomponenten v_1 und w_1 sind. Zu diesem Zweck müssen wir von $\hat{u}$ und $\check{u}$ verlangen, daß sie Lösungen der Differentialgleichungen

$$\frac{\partial u}{\partial y} - \frac{u}{\bar{R}-y} = -\frac{\partial u_0}{\partial y} + \frac{u_0}{\bar{R}-y}; \qquad \frac{\partial u}{\partial z} = 0 \tag{2.2}$$

sind, wodurch das Verschwinden der Koeffizienten von v_1 und w_1 erreicht wird.

Die allgemeine Lösung der Gleichungen (2.2) lautet:

$$\begin{aligned} \hat{u}(y,t) &= -u_0(y,t) + \frac{\hat{\alpha}(t)}{\bar{R}-y} \\ \check{u}(y,t) &= -u_0(y,t) + \frac{\check{\alpha}(t)}{\bar{R}-y}. \end{aligned} \tag{2.3}$$

Wir können nun über $\hat{\alpha}$ und $\check{\alpha}$ so verfügen, daß $\hat{u}$ und $\check{u}$ die folgenden, eine Ober- und Unterfunktion kennzeichnenden Ungleichungen befriedigen:

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &> \frac{1}{Re} L[\hat{u}] \\ \frac{\partial \check{u}}{\partial t} &< \frac{1}{Re} L[\check{u}] \end{aligned} \tag{2.4a}$$

und

$$\begin{aligned} &\hat{u}(y,t) > u^{(0)}(y,z), && \text{falls} \quad (y,z,t) \in G_i^*; \\ &\hat{u}(y,t) > 0, && \text{falls} \quad (y,z,t) \in S_i; \\ &\check{u}(y,t) < u^{(0)}(y,z), && \text{falls} \quad (y,z,t) \in G_i^*; \\ &\check{u}(y,t) < 0, && \text{falls} \quad (y,z,t) \in S_i, \qquad (i = 1, 2). \end{aligned} \tag{2.4b}$$

Falls die betrachtete Störung periodisch in z-Richtung ist, braucht man die Ungleichungen nur auf G_i^* und für ein Periodenintervall zu fordern.

* Im Zusammenhang mit der Entwicklung des Nagumo-Westphalschen Satzes und verwandter Sätze haben sich die Bezeichnungen „Ober-" und „Unterfunktion" eingebürgert. Ihre genaue Definition wird im Anhang wiedergegeben.

Für $\hat{\alpha}$ und $\check{\alpha}$ setzen wir an:

$$\begin{aligned}\hat{\alpha}(t) &= \hat{\alpha}^* e^{\hat{\beta} t}\\ \check{\alpha}(t) &= \check{\alpha}^* e^{\check{\beta} t}.\end{aligned} \tag{2.5}$$

Führt man diese Beziehungen in (2.3) und $\hat{u}$ bzw. $\check{u}$ in (2.4a) und (2.4b) ein, so zeigt sich, daß die genannten Ungleichungen erfüllt sind, wenn man

$$\begin{aligned}\hat{\alpha}^* &= \underset{(y,z,t)}{\mathrm{Max}}\begin{Bmatrix}(\bar{R}-y)\,[u^{(0)}(y,z)+u_0(y,t=0)]+\varepsilon;\\ [(\bar{R}-y)\,u_0(y,t)]_{(y,t)\in S_i}+\varepsilon; \quad \varepsilon\end{Bmatrix}\\ \check{\alpha}^* &= \underset{(y,z,t)}{\mathrm{Min}}\begin{Bmatrix}(\bar{R}-y)\,[u^{(0)}(y,z)+u_0(y,t=0)]-\varepsilon;\\ [(\bar{R}-y)\,u_0(y,t)]_{(y,t)\in S_i}-\varepsilon; \quad -\varepsilon\end{Bmatrix}\end{aligned} \tag{2.6}$$

$\varepsilon>0$, sonst beliebig und $\hat{\beta}>0$; $\check{\beta}>0$, sonst beliebig wählt.

Aus dem Nagumo-Westphalschen Satz (vgl. Anhang) folgt dann:

$$-u_0(y,t)+\frac{\check{\alpha}^* e^{\check{\beta} t}}{\bar{R}-y}<u_1(y,z,t)<-u_0(y,t)+\frac{\hat{\alpha}^* e^{\hat{\beta} t}}{\bar{R}-y}.$$

Läßt man nun in diesen Ungleichungen $\hat{\beta}$ und $\check{\beta}$ und in (2.6) ε gegen Null gehen, so folgt:

$$-u_0(y,t)+\frac{\check{\bar{\alpha}}}{\bar{R}-y}\leqq u_1(y,z,t)\leqq -u_0(y,t)+\frac{\hat{\bar{\alpha}}}{\bar{R}-y}, \tag{2.7}$$

mit

$$\begin{aligned}\hat{\bar{\alpha}} &= \underset{(y,z,t)}{\mathrm{Max}}\begin{Bmatrix}(\bar{R}-y)\,[u^{(0)}(y,z)+u_0(y,t=0)]\\ [(\bar{R}-y)\,u_0(y,t)]_{(y,t)\in S_i}; \quad 0\end{Bmatrix}\\ \check{\bar{\alpha}} &= \underset{(y,z,t)}{\mathrm{Min}}\begin{Bmatrix}(\bar{R}-y)\,[u^{(0)}(y,z)+u_0(y,t=0)];\\ [(\bar{R}-y)\,u_0(y,t)]_{(y,t)\in S_i}; \quad 0\end{Bmatrix}^{\star}.\end{aligned}$$

Die Beziehung (2.7) stellt nun die angekündigte Abschätzung der ersten Störungskomponente u_1 dar. Sie gilt, weil wir über T keine einschränkenden Annahmen gemacht haben, für jedes zeitliche Intervall $0\leqq t\leqq T$, und damit für alle Zeiten t: $0\leqq t\leqq\infty$. Dabei forderten wir von u_1 lediglich gewisse Stetigkeits- und Differenzierbarkeitseigenschaften und ferner, daß es im Unendlichen verschwindet oder in z-Richtung periodisch ist.

Ist die Störungskomponente u_1 zum Zeitpunkt $t=0$ ihrer Erzeugung klein gegenüber der Grundströmung $u_0(y, t=0)$, so liegen die hier abgeleiteten oberen und unteren Schranken von u_1 in der Größenordnung von $u_0(y, t)$. Ist dagegen $u^{(0)}(y, z)$ vergleichbar mit $u_0(y, t=0)$, so setzen sich die obere und die untere Schranke aus dem Maximal- bzw. Minimalwert der Grundströmung und den entsprechenden Werten der Anfangsverteilung $u^{(0)}(y, z)$ zusammen. Wenn $u_0(y, t)$ selbst für alle Zeiten beschränkt ist, was insbesondere für alle stationären Grundströmungen zutrifft, gilt dies nach (2.7) auch für die Störungskomponente u_1.

Zum Schlusse soll die Schärfe der Abschätzung (2.7) an Hand eines wichtigen Beispiels geprüft werden.

★ Um dieselbe Abschätzung für den Fall $\bar{R}=\infty$ durchführen zu können, hat man nur $\hat{\bar{\alpha}}/(\bar{R}-y)$ bzw. $\check{\bar{\alpha}}/(\bar{R}-y)$ durch zwei Konstanten $\hat{c}$ bzw. $\check{c}$ zu ersetzen.

u_0 sei die stationäre ebene Strömung in der Taylorschen Versuchsanordnung; ω_1 und ω_2 seien die Winkelgeschwindigkeiten des inneren und des äußeren Zylinders, R_1 und R_2 ihre Krümmungsradien. Als charakteristische Länge h wählen wir die Spaltbreite $h=R_2-R_1$, als Bezugsgeschwindigkeit die Umdrehungsgeschwindigkeit des inneren Zylinders $U_0=R_1\omega_1$.

Setzt man nun $\overline{R}=R_2/h=\overline{R}_2$; $\overline{R}-1=R_1/h=\overline{R}_1$,

so hat u_0 die folgende Gestalt:

$$u_0(y)=A(\overline{R}_2-y)+\frac{B}{\overline{R}_2-y},$$

mit

$$A=\frac{\overline{R}_1^2-(\omega_2/\omega_1)\,\overline{R}_2^2}{\overline{R}_1(1-2\overline{R}_2)}; \qquad B=\frac{\overline{R}_1\overline{R}_2^2(1-\omega_2/\omega_1)}{2\overline{R}_2-1}.$$

Um die Konstanten $\hat{\bar{\alpha}}$ und $\check{\bar{\alpha}}$ in den Gleichungen (2.7) zu berechnen, haben wir den in den geschweiften Klammern stehenden Ausdruck nach oben und unten abzuschätzen. Zu diesem Zweck betrachten wir die Funktion

$$\varphi(y)=(\overline{R}_2-y)\,u_0(y).$$

Sie wächst für $A<0$ monoton im Intervall $0\leqq y\leqq 1$, fällt monoton für $A>0$ in diesem Intervall und ist für $A=0$ konstant. Daher gilt:

$$\hat{\bar{\alpha}}=\begin{cases}\overline{R}_2\,u_M^{(0)}+\overline{R}_1\,u_0(1), & \text{für}\quad A<0,\\ \overline{R}_2\,u_M^{(0)}+\overline{R}_2\,u_0(0), & \text{für}\quad A\geqq 0;\end{cases}$$

$$\check{\bar{\alpha}}=\overline{R}_1\,u_m^{(0)};\quad A\ \text{beliebig},$$

mit

$$u_M^{(0)}=\operatorname*{Max}_{\overline{G}_1} u^{(0)}(y,z)$$
$$u_m^{(0)}=\operatorname*{Min}_{\overline{G}_1} u^{(0)}(y,z).$$

Wir schätzen nun noch die rechten Seiten der ersten Beziehung (2.7) nach oben ab. Für $A<0$ folgt: $B>\overline{R}_1$. Beachtet man noch, daß $u_0(1)=1$ und $u_0(0)=\overline{R}_2\,\omega_2/\overline{R}_1\,\omega_1$ ist, so zeigt man leicht, daß die Funktion

$$\varphi_1(y)=-u_0(y)+\frac{\overline{R}_1}{\overline{R}_2-y}$$

im Intervall $0\leqq y\leqq 1$ monoton fällt, so daß sich hieraus, zusammen mit (2.7), die Abschätzung ergibt:

$$\text{(2.8)}\quad -u_0(y)+\frac{\overline{R}_1}{\overline{R}_2-y}\,u_m^{(0)}\leqq u_1(y,z,t)\leqq\frac{\overline{R}_2}{\overline{R}_2-y}\,u_M^{(0)}+\frac{1}{\overline{R}_1\,\overline{R}_2\,\omega_1}\,[\overline{R}_1^2\,\omega_1-\overline{R}_2^2\,\omega_2],$$

für $A<0$, d.h. $\omega_2/\omega_1<\overline{R}_1^2/\overline{R}_2^2$.

Entsprechend kann für $A>0$ verfahren werden. In diesem Fall wächst nämlich die Funktion

$$\varphi_2(y)=-u_0(y)+\frac{\overline{R}_2}{\overline{R}_2-y}\,u_0(0)$$

monoton im Intervall $0\leqq y\leqq 1$. Daher gilt:

$$\text{(2.8a)}\quad -u_0(y)+\frac{\overline{R}_1}{\overline{R}_2-y}\,u_m^{(0)}\leqq u_1(y,z,t)\leqq\frac{\overline{R}_2}{\overline{R}_2-y}\,u_M^{(0)}+\frac{1}{\overline{R}_1^2\,\omega_1}\,[\overline{R}_2^2\,\omega_2-\overline{R}_1^2\,\omega_1],$$

für $A>0$, d.h. $\omega_2/\omega_1>\overline{R}_1^2/\overline{R}_2^2$.

Aus den oberen Schranken für u_1 in den Beziehungen (2.8) und (2.8a) erkennt man, daß diese Schranken immer kleiner werden, wenn A gegen Null geht. Bei gegebener Rotationsgeschwindigkeit des inneren Zylinders (gegebene Reynoldszahl in (2.1)) und geeigneter Wahl der Rotationsgeschwindigkeit des äußeren Zylinders (A hinreichend klein), kann man dann erreichen, daß sich die obere Schranke für u_1 um beliebig wenig von dem Term $\overline{R}_2 (\overline{R}_2 - y) u_M^{(0)}$ unterscheidet.

Endliche Störungskomponenten sind damit für genügend kleine A-Werte im wesentlichen durch den Maximalwert ihrer Anfangsverteilung nach oben beschränkt.

Für kleine Störungskomponenten lassen sich aber aus dieser Beziehung noch weitere Schlüsse ziehen. Da nämlich im Rahmen der linearisierten Stabilitätstheorie, die ja das Verhalten hinreichend kleiner Störungen angenähert richtig beschreibt, das Vorzeichen von u_1 nicht festgelegt ist (u_1 ist Lösung eines homogenen Randwertproblems), stellt die rechte Seite von (2.8) und (2.8a) sogar eine obere Schranke für $|u_1|$ dar. Liegt u_{1M}^{0} nun in dem durch die lineare Theorie erfaßbaren Bereich, so lehrt die Beziehung (2.8a), daß $|u_1|$ für genügend kleine A-Werte niemals aus diesem Bereich heraustreten kann. u_1 ist demnach sicher nicht angefacht. Die Grundströmung wird also gegen kleine Störungen immer stabiler, die kritische Reynoldszahl immer größer, wenn man sich dem Wert $A = 0$ nähert. Diese Aussage, die schon im Rayleighschen Kriterium enthalten ist, ergibt sich hier direkt aus den Abschätzungen.

Die vollständige Aussage dieses Kriteriums, daß nämlich die Grundströmung u_0 sogar für $A \leq 0$ gegen kleine Störungen stabil ist, läßt sich jedoch aus den obigen Abschätzungen nicht beweisen.

§ 3. Schranken für das zeitliche Anwachsen der Störungskomponenten v_1 und w_1

Da die Abschätzung der Störungskomponenten v_1 und w_1 mit Hilfe des im vorigen Paragraphen verwendeten Nagumo-Westphalschen Satzes nur grobe Schranken liefert – man kann lediglich zeigen, daß v_1 und w_1 höchstens linear mit der Zeit anwachsen –, werden wir in diesem Paragraphen zu einer anderen Methode greifen. Wir gehen von den Grundgleichungen über zu den Gleichungen für die gesamte, im durchströmten Raum vorhandene kinetische Energie der Komponenten v_1 und w_1. Auf dem Ergebnis des vorigen Paragraphen aufbauend, kann man dann mit einer schwachen und physikalisch belanglosen Einschränkung zeigen, daß dieser Energiebetrag für alle Zeiten unterhalb einer festen, von der Zeit unabhängigen Schranke bleibt, wenn nur die Grundströmung selbst beschränkt ist. Dies bedeutet aber die Beschränktheit für die Komponenten selbst.

Die Differentialgleichungen für die betrachteten Störungskomponenten erhält man wiederum durch Einsetzen von (1.1) in die Navier-Stokesschen Gleichungen. Wenn man beachtet, daß u_0 und p_0 diese Gleichungen schon befriedigen, folgt dann:

$$\begin{aligned}
&\frac{\partial v_1}{\partial t} - \frac{1}{Re} L[v_1] = -\frac{\partial p_1}{\partial y} - \frac{u_1(2u_0+u_1)}{\overline{R}-y} - \frac{\partial v_1}{\partial y} v_1 - \frac{\partial v_1}{\partial z} w_1,\\
&\frac{\partial w_1}{\partial t} - \frac{1}{Re} L_1[w_1] = -\frac{\partial p_1}{\partial z} - \frac{\partial w_1}{\partial y} v_1 - \frac{\partial w_1}{\partial z} w_1,\\
&\frac{\partial v_1}{\partial y} - \frac{v_1}{\overline{R}-y} + \frac{\partial w_1}{\partial z} = 0.
\end{aligned} \tag{3.1}$$

mit dem Operator:
$$L_1 \equiv L + \frac{1}{(\overline{R} - y)^2}.$$

Multipliziert man nun die erste Gleichung in (3.1) mit $(\overline{R} - y) v_1$ und die zweite Gleichung mit $(\overline{R} - y) w_1$, integriert man dann über den durchströmten Raum und addiert man schließlich die erhaltenen Ausdrücke, so ergibt sich, nach Durchführung mehrerer partieller Integrationen und unter Berücksichtigung der Kontinuitätsgleichung:

$$\begin{aligned}(3.2)\qquad &\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\overline{G}_1}(\overline{R}-y)\,(v_1^2+w_1^2)\,dy\,dz + \\ &+\frac{1}{Re}\cdot\int\limits_{\overline{G}_1}(\overline{R}-y)\left[\left(\frac{\partial v_1}{\partial y}\right)^2+\left(\frac{\partial v_1}{\partial z}\right)^2+\frac{v_1^2}{(\overline{R}-y)^2}+\left(\frac{\partial w_1}{\partial y}\right)^2+\left(\frac{\partial w_1}{\partial z}\right)^2\right]dy\,dz \\ &= -\int\limits_{\overline{G}_1} v_1 u_1 (2u_0+u_1)\,dy\,dz^{\star},\end{aligned}$$

dabei bedeutet $\overline{G}_1$ das durch seine Randpunkte abgeschlossene Gebiet G_1. Setzt man vorübergehend: $v_1 = \overline{v}_1 \sin z$ und $w_1 = \overline{w}_1 \sin z$, so folgt mit Hilfe dieser Substitution:

$$(3.3)\qquad \int\limits_{\overline{G}_1}(\overline{R}-y)\left(\frac{\partial v_1}{\partial z}\right)^2 dy\,dz = \int\limits_{\overline{G}_1}(\overline{R}-y)\,v_1^2\,dy\,dz + \int\limits_{\overline{G}_1}\left(\frac{\partial \overline{v}_1}{\partial z}\right)^2 \sin^2 z\,dy\,dz$$

und eine entsprechende Beziehung für w_1 und $\overline{w}_1$.

Wenn man nun (3.3) in (3.2) einsetzt, die rechte Seite von (3.3) mit der Cauchy-Schwarzschen Ungleichung nach oben und die linke Seite durch Vernachlässigung der Quadrate der Ableitungen sowie des Terms $v_1^2/(\overline{R} - y)^2$ nach unten abschätzt, so folgt:

$$\begin{aligned}(3.4)\qquad &\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\overline{G}_1}(\overline{R}-y)\,(v_1^2+w_1^2)\,dy\,dz + \frac{1}{Re}\int\limits_{\overline{G}_1}(\overline{R}-y)\,(v_1^2+w_1^2)\,dy\,dz \\ &\qquad \leqq \left\{\int\limits_{\overline{G}_1} v_1^2\,dy\,dz\cdot\int\limits_{\overline{G}_1} u_1^2(2u_0+u_1)^2\,dy\,dz\right\}^{\frac{1}{2}}.\end{aligned}$$

Führt man noch die Hilfsgrößen

$$[\varphi(t)]^2 = \int\limits_{\overline{G}_1}(\overline{R}-y)\,(v_1^2+w_1^2)\,dy\,dz; \qquad [\psi(t)]^2 = \int\limits_{\overline{G}_1} u_1^2(2u_0+u_1)^2\,dy\,dz$$

ein, so erhält man aus (3.4) die einfache Beziehung:

$$(3.5)\qquad \frac{d\varphi(t)}{dt} + \frac{1}{Re}\varphi(t) \leqq \psi(t).$$

Aus ihr läßt sich nun eine obere Schranke für φ ableiten, wenn nur ψ für alle Zeiten beschränkt ist. Hierfür müssen wir aber zunächst, wie im vorigen Paragraphen, voraussetzen, daß u_0 für $t \to \infty$ endlich bleibt. Ferner muß gefordert werden, daß das Integral:

$$(3.6)\qquad \int\limits_{\overline{G}_1}\overline{u}_1^2\,dy\,dz,$$

⋆ Falls die Störungskomponenten periodisch in z-Richtung sind, umfaßt der Integrationsbereich nur eine Periode.

mit

$$\bar{u}_1(y, z) = \operatorname*{Max}_{0 \leqq t \leqq \infty} \{|u_1(y, z, t)|\}$$

existiert. Nachdem wir aber im vorigen Paragraphen gezeigt haben, daß $|u_1|$ in jedem Punkt $(y, z, t) \in \bar{Z}_1$ beschränkt ist, erscheint diese Annahme vom physikalischen Standpunkt aus nicht einschneidend. Betrachtet man etwa den Taylorschen Fall der koaxialen Zylinder, so ist die obige Forderung schon dann erfüllt, wenn man von der idealisierten Vorstellung unendlich langer Zylinder abgeht und endliche Zylinderhöhen annimmt.

Nach den Ungleichungen (2.7) läßt sich $|u_0 + u_1|$ durch $\operatorname{Max}\{1/(\bar{R} - y)(\hat{\bar{\alpha}}, |\check{\bar{\alpha}}|)\}$ nach oben abschätzen, woraus sich, zusammen mit der Annahme (3.6), die folgende Ungleichung ergibt:

$$(3.7) \quad |\psi(t)| \leqq \left\{ \int_{\bar{G}_1} \bar{u}_1^2 \, d y \, d z \right\}^{\frac{1}{2}} \cdot \operatorname*{Max}_{\bar{Z}_1} \{|u_0| + |u_0 + u_1|\} \leqq M; \qquad (M = \text{konst.}).$$

Nun folgt aber aus (3.5) und (3.7):

$$(3.8) \qquad \varphi(t) \leqq N e^{(-1/Re)t} + Re\, M,$$

wobei

$$N \geqq \left\{ \int_{\bar{G}_1} (\bar{R} - y) \left[(v_1^{(0)})^2 + (w_1^{(0)})^2\right] d y \, d z \right\}^{\frac{1}{2}} - Re\, M$$

ist. Die Beziehung (3.8) ist aber unsere Behauptung, daß nämlich die Komponenten aller betrachteten Störungen für alle Zeiten unterhalb einer festen Schranke bleiben*.

Um analoge Abschätzungen für $\bar{R} = \infty$ zu erhalten, hat man die Gleichung (3.1) für $\bar{R} = \infty$ mit v_1 und w_1 zu multiplizieren, über $\bar{G}_2$ zu integrieren und die so erhaltenen Beziehungen zu addieren. Die Beschränktheit der Störungskomponenten v_1 und w_1 ergibt sich dann, wie dies oben im Falle der zylindrisch gekrümmten Wand dargelegt wurde.

§ 4. Ein allgemeines Stabilitätskriterium

Die im vorigen Paragraphen benutzte Methode soll zum Schluß zur Herleitung des in der Einleitung angekündigten Stabilitätskriteriums verwendet werden. Wir schränken dabei die Allgemeinheit der Aussage wiederum nur durch die Forderung ein, daß die Grundströmung und sämtliche Störungskomponenten von der Hauptströmungsrichtung unabhängig sind.

Die Beziehung (2.1) wird jetzt noch mit $(\bar{R} - y)\, u_1$ multipliziert, sodann über den Bereich $\bar{G}_1$ integriert, und endlich wird die so erhaltene Gleichung zur Gleichung (3.2) addiert. Benutzt man wiederum (3.3) und vernachlässigt danach alle Quadrate der Ableitungen, so ergibt sich mit Hilfe der Cauchy-Schwarzschen Ungleichung:

$$(4.1) \qquad \frac{1}{2} \frac{d[\bar{\varphi}(t)]^2}{dt} + \frac{1}{Re} [\bar{\varphi}(t)]^2 \leqq C [\bar{\varphi}(t)]^2,$$

---

* Vom rein mathematischen Standpunkt aus müßte diese Aussage etwas modifiziert werden. Aus der Beschränktheit der Gesamtenergie φ folgt nämlich lediglich, daß die Funktionen v_1 und w_1 „fast überall" beschränkt sind.

mit

$$[\bar{\varphi}(t)]^2 = \int_{\bar{G}_1} (\bar{R} - y)\,(u_1^2 + v_1^2 + w_1^2)\,d\,y\,d\,z$$

und

$$C = \frac{1}{2} \operatorname*{Max}_{\bar{Z}_1} \left\{ \left| \frac{\partial u_0}{\partial y} + \frac{u_0}{\bar{R} - y} \right| \right\}.$$

Aus (4.1) folgt nun aber sofort, daß für $Re < 1/C$

$$\lim_{t \to \infty} |\bar{\varphi}(t)|^2 = 0$$

gilt. Mithin sind alle infinitesimalen und alle endlichen Störungen im Bereich $Re < 1/C$ gedämpft, wenn nur die Grundströmung und die Störungen von der Hauptströmungsrichtung unabhängig sind.

Eine einfache Rechnung ergibt im Taylor-Fall:

$$C = \frac{|B|}{\bar{R}_1^2}.$$

Alle stationären Strömungen in der Taylorschen Versuchsanordnung sind daher im Bereich

$$Re < Re_0 = \frac{\bar{R}_1 (\bar{R}_2^2 - \bar{R}_2^2)}{\bar{R}_2^2 |1 - \omega_2/\omega_1|} = \frac{1}{|1 - \omega_2/\omega_1|} \left\{ 2 - \frac{3}{R_2/h} + \frac{1}{(R_2/h)^2} \right\} \tag{4.2}$$

gegenüber von x unabhängigen Störungen beliebiger Größe stabil. (Die Reynoldszahl Re wird hierbei wie in §2 mit der Rotationsgeschwindigkeit des inneren Zylinders $R_1 \omega_1$ und mit der Spaltbreite $h = R_2 - R_1$ gebildet; $Re = R_1 \omega_1 h/\nu$.)

Ein Vergleich der in (4.2) abgeleiteten Stabilitätsgrenze Re_0 mit der von J. Serrin in [1] angegebenen (wir nennen sie Re_1) fällt zugunsten der letzteren aus. Serrins Ergebnis lautet nämlich, in den hier verwendeten Symbolen geschrieben:

$$Re_1 = \frac{\pi^2 (\bar{R}_2^2 - R_1^2)}{\bar{R}_1 \bar{R}_2^2 [\log (\bar{R}_2 | \bar{R}_1)]^2 |1 - \omega_2/\omega_1|}.$$

Es gilt stets $\pi^2 Re_0 \leq Re_1$, wobei das Gleichheitszeichen im Grenzfall $R_1/h \to \infty$ angenommen wird. Da Serrins Stabilitätsgrenze darüber hinaus auch für x-abhängige Störungen gilt, ist die Aussage der Beziehung (4.2) schon völlig im Serrinschen Ergebnis enthalten.

Als zweites Beispiel werde die durch ein Druckgefälle erzeugte stationäre Schichtenströmung zwischen zwei koaxialen Zylinderwänden untersucht. Die Spaltbreite $h = R_2 - R_1$ sei wiederum als charakteristische Länge, die Maximalgeschwindigkeit der Grundströmung u_{0M} als Bezugsgeschwindigkeit gewählt. u_0 hat dann die Gestalt:

$$u_0(y) = \gamma \left| a(\bar{R}_2 - y) + \frac{b}{\bar{R}_2 - y} - (\bar{R}_2 - y) \log (\bar{R}_2 - y) \right|,$$

mit

$$a = \frac{\bar{R}_2^2 \log \bar{R}_2 - \bar{R}_1^2 \log \bar{R}_1}{\bar{R}_2^2 - \bar{R}_1^2}$$

$$b = -\frac{\bar{R}_1^2 \bar{R}_2^2}{\bar{R}_2^2 - \bar{R}_1^2} \log \frac{\bar{R}_2}{\bar{R}_1}$$

$$\gamma = \frac{\bar{R}_0}{2b + \bar{R}_0^2},$$

dabei ist $\overline{R}_0 = \overline{R}_2 - y_0$, und y_0 ist durch die Gleichung $u_0(y_0) = u_{0M}$ bestimmt. Der oben abgeleitete Stabilitätsbereich $Re < 1/C$ lautet in diesem Fall:

$$Re < Re_0 = \frac{2\overline{R}_1^2 |2b + \overline{R}_0^2|}{\overline{R}_0 |2b + \overline{R}_1^2|}. \tag{4.3}$$

Da sich $2b/\overline{R}_0^2$ für großes $\overline{R}_1$ wie $2b/\overline{R}_1^2$ verhält, ergibt sich an Stelle des Kriteriums (4.3) die einfachere Beziehung:

$$Re < 2\overline{R}_0. \tag{4.3 a}$$

Der Stabilitätsbereich der betrachteten Kanalströmung wächst also für hinreichend großes $\overline{R}_1$ monoton mit wachsendem $\overline{R}_0$. Die ebene Kanalströmung selbst ist völlig stabil gegenüber Störungen des hier betrachteten Typs.

Ein Vergleich mit der von J. Serrin für beliebige Grundströmungen angegebenen Stabilitätsgrenze $Re = 5{,}71$ — sie wird in der zitierten Arbeit allerdings nur für beschränkte Bereiche bewiesen — zeigt, daß die Beziehungen (4.3) und (4.3 a) für genügend großes $\overline{R}_1$ größere Stabilitätsbereiche liefern. Dies war zu erwarten; denn die ebene Kanalströmung wird bekanntlich schon gegenüber einer speziellen Störungsart der bei Serrin zugelassenen allgemeinen Störungen, nämlich der Tollmien-Schlichting-Wellen, bei endlichen Reynoldszahlen instabil.

Im Falle der ebenen Platte ($\overline{R} = \infty$) lautet das Stabilitätskriterium entsprechend: Für $Re < 1/C$, mit $C = \operatorname{Max}_{Z_s} \left\{ \frac{1}{2} \left| \frac{\partial u_0}{\partial y} \right| \right\}$ sind alle infinitesimalen und alle endlichen, von x unabhängigen Störungen gedämpft.

§ 5. Anhang

Der Nagumo-Westphalsche Satz. Wir haben in der vorliegenden Arbeit den Nagumo-Westphalschen Satz an einer wesentlichen Stelle, nämlich bei der Abschätzung der Störungskomponente u_1 benutzt. Desgleichen verwendeten wir in diesem Zusammenhang die Bezeichnungen „Ober"- und „Unterfunktion". Diese Bezeichnungen sowie die Aussage des Satzes sollen hier im Anhang kurz wiedergegeben werden. Bei der Formulierung der Voraussetzungen beschränken wir uns auf die in dieser Arbeit behandelten speziellen Strömungsgebiete und Randbedingungen und verweisen im übrigen, was den Beweis und die Erweiterung auf allgemeinere Bereiche anbetrifft, auf die Arbeit von K. Nickel [8].

Im zweidimensionalen euklidischen Raum, in dem wir die kartesischen Koordinaten y, z eingeführt denken, sei ein Gebiet $G = \{0 < y < 1;\ -1 < z < +1\}$ gegeben; ferner ein Intervall $I = \{0 < t < T\}$. Die Gesamtheit der Punkte (y, z, t) mit $(y, z) \in G$ und $t \in I$ bilden das Innere eines Zylinders Z, dessen Mantelfläche wir mit S und dessen Grundfläche (Schnitt mit der Ebene $t = 0$) wir mit G^* bezeichnen. $\overline{Z}$ sei derjenige abgeschlossene Bereich, der aus Z durch Hinzunahme sämtlicher Randpunkte hervorgeht. Ferner sei die Funktion $u(y, z, t)$ Lösung des folgenden Randwertproblems:

$$\begin{aligned} &\frac{\partial u}{\partial t} = F\left(t, y, z; u; \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}; \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}\right), \\ &u(y, z, t = 0) = u^{(0)}(y, z); \\ &u(y, z, t) = 0, \quad \text{falls} \quad (y, z, t) \in S \end{aligned} \tag{5.1}$$

u existiere und sei stetig in $\bar{Z}$. Die ersten und zweiten Ableitungen von u nach y und z und die erste Ableitung von u nach t sollen in Z existieren und in $Z \cup G^* \cup S$ stetig sein.

Die Funktion F sei eindeutig definiert und beschränkt für $(y, z, t) \in \bar{Z}$ und für beliebige reelle Werte der restlichen Variablen. Darüber hinaus soll F in den zwei letzten Variablen schwach monoton anwachsen. Es sei also

$$F(t, z, y; q_1, q_2; q_{11}, q_{21}) \leqq F(t, y, z; q_1, q_2; q_{12}, q_{22})\,^\star$$

falls $q_{11} \leqq q_{12}$ und $q_{21} \leqq q_{22}$ gilt.

$\hat{u}$ bzw. $\check{u}$ heißen Ober- bzw. Unterfunktion von u, wenn sie allen an u gestellten Stetigkeits- und Differenzierbarkeitsforderungen genügen und die folgenden Ungleichungen erfüllen:

$$\frac{\partial \hat{u}}{\partial t} > F\left(t, y, z; \frac{\partial \hat{u}}{\partial y}, \frac{\partial \hat{u}}{\partial z}; \frac{\partial^2 \hat{u}}{\partial y^2}, \frac{\partial^2 \hat{u}}{\partial z^2}\right),$$

$$\hat{u}(y, z, t=0) > u^{(0)}(y, z); \tag{5.1a}$$

$$\hat{u}(y, z, t) > 0, \quad \text{falls} \quad (y, z, t) \in S$$

bzw.

$$\frac{\partial \check{u}}{\partial t} < F\left(t, y, z; \frac{\partial \check{u}}{\partial y}, \frac{\partial \check{u}}{\partial z}; \frac{\partial^2 \check{u}}{\partial y^2}, \frac{\partial^2 \check{u}}{\partial z^2}\right),$$

$$\check{u}(y, z, t=0) < u^{(0)}(y, z); \tag{5.1b}$$

$$\check{u}(y, z, t) < 0, \quad \text{falls} \quad (y, z, t) \in S.$$

Es gilt dann der

Satz von Nagumo-Westphal. Ist die Funktion u eine Lösung des Randwertproblems (5.1), die sämtliche oben angeführten Stetigkeits- und Differenzierbarkeitseigenschaften besitzt, und ist $\hat{u}$ eine Oberfunktion und $\check{u}$ eine Unterfunktion von u, dann gilt im ganzen Bereich $\bar{Z}$:

$$\check{u}(y, z, t) < u(y, z, t) < \hat{u}(y, z, t).$$

Die Erweiterung dieses Satzes auf nichtbeschränkte Gebiete kann nach K. NICKEL [8] in einfacher Weise geschehen. Wir führen den Gedankengang an Hand der in §1 definierten unbeschränkten Gebiete G_i durch. Zunächst werden diese Gebiete topologisch auf G abgebildet. Dies leisten die Transformationen:

$$y = a\, y^*; \qquad z = \frac{z^*}{1 - z^{*2}}; \qquad (-1 < z^* < +1), \quad \text{für} \quad i = 1,$$

$$y = \frac{y^*}{1 - y}; \qquad z = \frac{z^*}{1 - z^{*2}}; \qquad (-1 < z^* < +1), \quad \text{für} \quad i = 2.$$

Die Randpunkte von G_i gehen dabei in Randpunkte von G über; daneben enthält aber der Rand von G noch weitere Punkte (etwa $z^* = \pm 1$), die als Bildpunkte der unendlich fernen Punkte von G_i aufgefaßt werden.

Durch die Transformationen sind nun auch für das Gebiet G_i und damit auch für Z_i alle Randpunkte durch ihre topologischen Bilder eindeutig erklärt. Definiert man nun noch „Randwerte" einer Funktion in den unendlich fernen Punkten durch die Werte der Funktion in den zugeordneten Punkten, ebenso die

* Man erkennt leicht, daß die rechte Seite der Gleichung (2.1) die hier an F gestellten Forderungen erfüllt.

Begriffe: Konvergenz einer Punktfolge, Stetigkeit und Differenzierbarkeit einer Funktion im Bereich G_i durch die entsprechenden Eigenschaften in G, dann läßt sich die Aussage des Nagumo-Westphalschen Satzes auch für die hier verwendeten nichtbeschränkten Grundgebiete beweisen.

Zusammenfassung

In dieser Arbeit wird das Verhalten dreidimensionaler Störungen einer stationären oder instationären Schichtenströmung längs einer zylindrisch gekrümmten Wand untersucht. Grundströmung und Störungen sollen von der Hauptströmungsrichtung unabhängig sein, unterliegen aber sonst keinerlei physikalisch einschränkenden Voraussetzungen; insbesondere werden Störungen beliebiger Größe zugelassen, wobei die bekannten Taylor-Görtler-Wirbel als Spezialfall in diesen Untersuchungen mit enthalten sind. Es werden Schranken angegeben, über die hinaus die Störungskomponenten niemals anwachsen können. Diese Schranken bleiben für alle Zeiten endlich, wenn die Grundströmungsgeschwindigkeit selbst endlich bleibt.

Ferner wird gezeigt, daß alle betrachteten infinitesimalen oder endlichen Störungen gedämpft sind, falls die Reynoldszahl einen gewissen, wiederum von der Grundströmung abhängigen und oben angegebenen Wert nicht überschreitet.

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On the Integration of the Equations of Motion in the Classical Theory of Elasticity

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Contents

Page
1. Introduction 34
2. DUHEM's Proof of CLEBSCH's Theorem 37
3. LAMÉ's Solution in the Equilibrium Case 39
4. Connection between LAMÉ's and IACOVACHE's Solution 43
5. Axisymmetric and Two-dimensional Solutions of the Equations of Motion . 45
References 49

1. Introduction

The present paper is concerned with the theory of integration of the displacement-equations of motion in the linear theory of homogeneous and isotropic elastic solids. These equations, which in their general form originated with CAUCHY [*1*][1] (1828), in vector notation, and for the case of vanishing body forces, appear as

$$\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu} \nabla \nabla \cdot \boldsymbol{u} = \frac{\varrho}{\mu} \frac{\partial^2 \boldsymbol{u}}{\partial t^2}. \tag{1}$$

Here $\boldsymbol{u}(x, t)$ is the displacement vector[2] — x standing for the triplet of rectangular cartesian coordinates (x_1, x_2, x_3) — t denotes the time, the constants ϱ, μ, and ν designate the mass density, the shear modulus, and POISSON's ratio, respectively, while ∇ is the usual del-operator.

In his mémoire on the equilibrium and motion of elastic bodies, POISSON [*2*] (1829) introduced a class of particular solutions to (1) that is obtained by assuming the displacement vector to be the gradient of a scalar potential. In a subsequent addition to the foregoing paper, POISSON [*3*] (1829) proved[3] what appears to be the earliest theorem concerning the general integral of the equations of motion. He showed that every (sufficiently regular) solution of (1) admits the representation

$$\boldsymbol{u}(x,t) = \boldsymbol{u}'(x,t) + \boldsymbol{u}''(x,t), \quad \boldsymbol{u}' = \nabla\varphi, \quad \nabla \cdot \boldsymbol{u}'' = 0, \tag{2}$$

where $\varphi(x, t)$ and $\boldsymbol{u}''(x, t)$ satisfy the wave equations

$$\Box_1^2 \varphi = 0, \quad \Box_2^2 \boldsymbol{u}'' = 0, \tag{3}$$

---

[1] Numbers in brackets refer to the list of references at the end of the paper.

[2] Throughout this paper letters in boldface represent vectors. The symbols "$\cdot$" and "$\times$" are used to indicate scalar and vector multiplication, respectively.

[3] This proof is summarized by TODHUNTER & PEARSON [*4*], Vol. 1, p. 273 *et seq.*

in which

$$\square_n^2 = \nabla^2 - \frac{1}{c_n^2}\frac{\partial^2}{\partial t^2} \qquad (n=1,2),$$
$$c_1^2 = \frac{2(1-\nu)\,\mu}{(1-2\nu)\,\varrho}, \qquad c_2^2 = \frac{\mu}{\varrho}. \tag{4}$$

Poisson thus established the fundamental fact that the complete solution of (1) may be expressed as a superposition of an irrotational and an equivoluminal motion, associated with the respective wave velocities c_1 and c_2. The results contained in [2], [3] rest on the molecular hypothesis of elastic action which, in the case of the isotropic medium, implies the existence of but a single independent elastic constant and is equivalent to choosing $\nu=\frac{1}{4}$ in (1). Consequently, the wave velocities c_1, c_2 defined in (4) were found by Poisson to obey the relation $c_1^2=3c_2^2$. His proof [3] of the preceding theorem, however, in no way depends on this particular choice of the parameter ν.

Stokes [5] (1851), in dealing with the general isotropic medium of unlimited extent, approached the integration of the equations of motion by deducing from (1) that the dilatation ϑ and the rotation $\boldsymbol{\omega}$, which are given by

$$\vartheta(x,t) = \nabla\cdot\boldsymbol{u}, \qquad \boldsymbol{\omega}(x,t) = \nabla\times\boldsymbol{u}, \tag{5}$$

must satisfy the wave equations

$$\square_1^2\vartheta = 0, \qquad \square_2^2\boldsymbol{\omega} = 0. \tag{6}$$

He then made use of an integral representation for the displacement $\boldsymbol{u}(x,t)$ in terms of ϑ and $\boldsymbol{\omega}$.

The general solution (2), (3), due to Poisson, does not involve the vector potential appropriate to the solenoidal displacement component $\boldsymbol{u}''$. A solution of (1) which employs both a scalar and a vector potential apparently made its first appearance[4] in Lamé's [6] (1852) treatise. Lamé noted that every vector field of the form

$$\boldsymbol{u}(x,t) = \nabla\varphi + \nabla\times\boldsymbol{\psi} \tag{7}$$

meets (1) provided $\varphi(x,t)$ and $\boldsymbol{\psi}(x,t)$ are particular solutions of

$$\square_1^2\varphi = 0, \qquad \square_2^2\boldsymbol{\psi} = 0, \tag{8}$$

as is readily confirmed by substitution.

The question as to the completeness of the solution (7), (8) was raised by Clebsch [7] (1863), who asserted that every solution of the equations of motion (1) in turn admits the representation (7), (8). With a view toward proving this theorem, Clebsch independently demonstrated the completeness of the solution (2), (3) by an argument which is somewhat less direct than that originally produced by Poisson [3], but without specializing the value of ν; he subsequently assumed without proof that every $\boldsymbol{u}''(x,t)$ satisfying the last of (2) and the second of (3) may be represented by

$$\boldsymbol{u}''(x,t) = \nabla\times\boldsymbol{\psi}, \qquad \square_2^2\boldsymbol{\psi} = 0. \tag{9}$$

Another inconclusive proof of Clebsch's theorem was given later by Kelvin [8][5] (1884).

[4] See Todhunter & Pearson [4], Vol. 2, Pt. 2, p. 99.

[5] See Lecture IV, p. 41 *et seq.* This lecture was delivered in 1884 although [8] was published in 1904.

Duhem [9] (1898), after expressing reservations about Clebsch's [7] completeness proof[6], supplied a rigorous proof of the theorem previously announced in [7]. In a second paper, Duhem [10] (1900) extended the result established in [9] to more general systems of linear partial differential equations.

Unfortunately, Duhem's [9] original paper, which is not readily accessible, appears to be little known, and treatises on the theory of elasticity usually refrain from discussing the generality of Lamé's solution (7), (8), or take its completeness for granted. In contrast, Sneddon & Berry [11][7] (1958) state that Lamé's solution is incomplete if the region occupied by the medium has a boundary. A proof of Clebsch's theorem appearing in Pearson's [12] (1959) book is open to objections, as is a brief argument advanced in this connection by Bishop [13] (1953).

In view of the preceding remarks, and because of the far-reaching importance of Lamé's solution in elastokinetics, it seems appropriate to include Duhem's proof of Clebsch's theorem in this paper (Section 2). At the same time, the present vectorial version of the proof is apt to bring out more clearly the essential simplicity of Duhem's reasoning.

In Section 3 we explore the significance of Lamé's solution within the equilibrium theory. While there is nothing in Duhem's completeness proof that impairs its validity when $\boldsymbol{u}$ happens to be independent of the time, (7) and (8), with φ and $\boldsymbol{\psi}$ functions of position alone, clearly cannot be the general solution of (1) in the equilibrium case: for in this event (8) would degenerate into Laplace equations and hence, by (7), every equilibrium solution $\boldsymbol{u}(x)$ of (1) would have to be harmonic (rather than merely biharmonic). This apparent contradiction is resolved by observing that even in the equilibrium case the potentials φ and $\boldsymbol{\psi}$, in general, remain time-dependent. Thus every *equilibrium* field $\boldsymbol{u}(x)$ conforming to (1) may be represented as the sum of an irrotational and an equivoluminal *motion*, the time-dependence of which cancels upon addition. In the section here referred to we determine the explicit manner in which $\varphi(x, t)$ and $\boldsymbol{\psi}(x, t)$ involve t when $\boldsymbol{u}=\boldsymbol{u}(x)$ by establishing a connection between the Lamé potentials φ, $\boldsymbol{\psi}$ and the Papkovich-Neuber [14], [15] potentials of the equilibrium theory. The resolution of an equilibrium solution into irrotational and equivoluminal dynamic components is illustrated with a specific example. This type of resolution has an analogue in the theory of the repeated wave equation, as is made clear at the end of Section 3.

Section 4 is concerned primarily with the relation between Lamé's solution (7), (8) and a solution of (1) discovered by Iacovache [16] on the basis of a formal operational scheme due to G. C. Moisil [17]. We show that Iacovache's solution, which is the dynamic counterpart of Galerkin's [18] general equilibrium solution, may be reduced to Lamé's solution with the aid of a simple transformation. Since the completeness of Iacovache's result was established in a previous paper [19], this transformation supplies an alternative proof of Clebsch's completeness theorem for Lamé's solution. At the end of Section 4, a recent general solution due to Noll [20] of the equations of motion for steady-state oscillations

[6] "Malheureusement, la démonstration donnée par Clebsch peut laisser place à quelques doutes."

[7] See p. 109.

is identified as a special case of IACOVACHE's solution. Finally, Section 5 is devoted to general solutions of the equations of motion in the special instances of rotational symmetry and of plane strain (or generalized plane stress).

For the sake of focusing on essentials, the body forces are supposed to vanish throughout the succeeding analysis; the extension of the results to the case in which body forces are present offers no difficulties. Also, in order to avoid a steady repetition of tedious regularity assumptions, all such hypotheses are omitted from the statement of the theorems to be discussed. Suffice it to say that the region D under consideration need not be bounded or simply connected. If D is bounded, we require merely that $\boldsymbol{u}(x, t)$, together with its space and time derivatives of the first and second order, be continuous for (x_1, x_2, x_3) in D, $t_1<t<t_2$. If D is not bounded, $\boldsymbol{u}(x, t)$ must, in addition, be subjected to suitable regularity conditions at infinity.

2. Duhem's proof of Clebsch's Theorem

The completeness theorem announced by CLEBSCH [7] may be stated as follows: *Let $\boldsymbol{u}(x, t)$ be a particular solution of* (1) *in a region of space D and for $t_1<t<t_2$. Then there exists a scalar function $\varphi(x, t)$ and a vector function $\boldsymbol{\psi}(x, t)$ such that $\boldsymbol{u}(x, t)$ is represented by* (7), *with*

$$\nabla\cdot\boldsymbol{\psi}=0, \tag{10}$$

and φ, $\boldsymbol{\psi}$ satisfy (8) *in which the wave operators $\Box_n^2$ $(n=1, 2)$ are given by* (4).

We now summarize, in vectorial notation, DUHEM's [9] proof of the preceding theorem. Let $\boldsymbol{W}(x, t)$ be the Newtonian potential defined by

$$\boldsymbol{W}(x, t) = -\frac{1}{4\pi}\int\limits_D \frac{\boldsymbol{u}(\xi, t)\,dv}{R(x, \xi)}, \tag{11}$$

where $R(x, \xi)$ is the distance between two points of D with coordinates (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3), respectively. Define $U(x, t)$ and $\boldsymbol{V}(x, t)$ by

$$U=\nabla\cdot\boldsymbol{W}, \qquad \boldsymbol{V}=-\nabla\times\boldsymbol{W}, \tag{12}$$

note from (11) that throughout D

$$\nabla^2\boldsymbol{W}=\boldsymbol{u}, \tag{13}$$

and recall the general identity

$$\nabla^2\boldsymbol{W}=\nabla\nabla\cdot\boldsymbol{W}-\nabla\times\nabla\times\boldsymbol{W}. \tag{14}$$

From (12), (13), (14) follows

$$\boldsymbol{u}=\nabla U+\nabla\times\boldsymbol{V}, \qquad \nabla\cdot\boldsymbol{V}=0, \tag{15}$$

which is merely the usual Helmholtz representation of the vector field $\boldsymbol{u}(x, t)$.

Substitution for $\boldsymbol{u}$ from the first of (15) into (1), and recourse to (4), yield

$$c_1^2\nabla\Box_1^2 U+c_2^2\nabla\times\Box_2^2\boldsymbol{V}=0. \tag{16}$$

Operating on (16) with the divergence and the curl, respectively, one obtains with the aid of the identity (14) (applied to $\boldsymbol{V}$), in view of the second of (15),

$$\nabla^2\Box_1^2 U=0, \qquad \nabla^2\Box_2^2\boldsymbol{V}=0. \tag{17}$$

Thus

$$\Box_1^2 U = a(x, t), \qquad \Box_2^2 \boldsymbol{V} = \boldsymbol{b}(x, t), \tag{18}$$

where

$$\nabla^2 a = 0, \qquad \nabla^2 \boldsymbol{b} = 0, \qquad \nabla \cdot \boldsymbol{b} = 0, \tag{19}$$

the last of (19) being a consequence of the second of (15) and the second of (18).

Now, define $A(x, t)$ and $\boldsymbol{B}(x, t)$ through

$$A = \frac{1}{c_1^2} \int_{t_0}^{t} \int_{t_0}^{\tau} a(x, \lambda)\, d\lambda\, d\tau, \qquad \boldsymbol{B} = \frac{1}{c_2^2} \int_{t_0}^{t} \int_{t_0}^{\tau} \boldsymbol{b}(x, \lambda)\, d\lambda\, d\tau, \tag{20}$$

where $t_1 < t_0 < t_2$, so that

$$a = \frac{1}{c_1^2} \frac{\partial^2 A}{\partial t^2}, \qquad \boldsymbol{b} = \frac{1}{c_2^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{21}$$

and, because of (19),

$$\nabla^2 A = 0, \qquad \nabla^2 \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{22}$$

Next, let $U_1(x, t)$ and $\boldsymbol{\psi}(x, t)$ be defined by

$$U_1 = U + A, \qquad \boldsymbol{\psi} = \boldsymbol{V} + \boldsymbol{B}. \tag{23}$$

Substituting for U, $\boldsymbol{V}$ from (23) into (18), and taking account of (21), (22), one arrives at

$$\Box_1^2 U_1 = 0, \qquad \Box_2^2 \boldsymbol{\psi} = 0. \tag{24}$$

By (15) and (23),

$$\boldsymbol{u} = \nabla U_1 + \nabla \times \boldsymbol{\psi} + \boldsymbol{u}^*, \tag{25}$$

provided

$$\boldsymbol{u}^* = -\nabla A - \nabla \times \boldsymbol{B}. \tag{26}$$

Equations (26), (22), together with the identity (14) applied to $\boldsymbol{B}$, imply

$$\nabla \cdot \boldsymbol{u}^* = 0, \qquad \nabla \times \boldsymbol{u}^* = 0. \tag{27}$$

Therefore, there exists a function $U_2(x, t)$ such that

$$\boldsymbol{u}^* = \nabla U_2, \qquad \nabla^2 U_2 = 0, \tag{28}$$

and the first of (28), in accordance with (25), gives

$$\boldsymbol{u} = \nabla U_1 + \nabla U_2 + \nabla \times \boldsymbol{\psi}. \tag{29}$$

Substituting from (29) into (1) and bearing in mind (4), one obtains in view of (24) and the second of (28)

$$\nabla \frac{\partial^2 U_2}{\partial t^2} = 0, \tag{30}$$

whence

$$U_2 = \alpha(t) + t\beta(x) + \gamma(x). \tag{31}$$

By (31) and the second of (28),

$$\nabla^2 \beta = 0, \qquad \nabla^2 \gamma = 0. \tag{32}$$

Finally, define $\varphi(x, t)$ through

$$\varphi = U_1(x, t) + U_2(x, t) - \alpha(t). \tag{33}$$

According to (33), (29), in conjunction with the second of (15), the last of (22), and the second of (23),

$$\boldsymbol{u} = \nabla\varphi + \nabla\times\boldsymbol{\psi}, \quad \nabla\cdot\boldsymbol{\psi} = 0, \tag{34}$$

while from (24), (31), (32), and (33),

$$\Box_1^2\varphi = 0, \quad \Box_2^2\boldsymbol{\psi} = 0. \tag{35}$$

This completes the proof of the theorem. It should be noted that $\boldsymbol{u}(x, t)$, given by the first of (34), satisfies (1) provided (35) hold even if $\boldsymbol{\psi}(x, t)$ is not solenoidal, as may be confirmed directly.

DUHEM's argument is easily extended to accomodate body forces and a time-dependent temperature field. Such an extension, in particular, assures the completeness of a general solution to the coupled thermoelastic field equations employed recently by DERESIEWICZ [*33*].

3. Lamé's Solution in the Equilibrium Case

Since the foregoing proof of CLEBSCH's theorem does not preclude the possibility that $\boldsymbol{u}$ is independent of the time t, LAMÉ's solution (7), (8) of the equations of motion (1) clearly remains complete in the special case of equilibrium. As pointed out in the Introduction, however, the potentials φ and $\boldsymbol{\psi}$ are bound to remain time-dependent also in this event unless $\boldsymbol{u}(x)$ happens to be harmonic[8]. We now examine the manner in which LAMÉ's potentials involve the time when $\boldsymbol{u}$ is a solution of the equilibrium equations and establish the relation which φ and $\boldsymbol{\psi}$ bear to the PAPKOVICH-NEUBER potentials of the equilibrium theory.

To this end we recall that PAPKOVICH's [*14*] solution, which was independently rediscovered by NEUBER [*15*], has the form

$$\boldsymbol{u}(x) = \nabla(\Phi + \boldsymbol{x}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\boldsymbol{\Psi}, \tag{36}$$

where $\boldsymbol{x}$ is the position vector with components (x_1, x_2, x_3), and $\Phi(x)$, as well as $\boldsymbol{\Psi}(x)$, are solutions of the Laplace equations

$$\nabla^2\Phi = 0, \quad \nabla^2\boldsymbol{\Psi} = 0. \tag{37}$$

The completeness of the equilibrium solution (36), (37) was established first in [*14*]. A more direct completeness proof was supplied later by MINDLIN [*21*][9].

We now prove the following theorem: *Suppose an equilibrium solution* $\boldsymbol{u}(x)$ *of* (1) *is generated, in the sense of* (36), (37), *by the Papkovich-Neuber potentials* $\Phi(x)$ *and* $\boldsymbol{\Psi}(x)$. *Then it is also generated, in the sense of* (7), (8), *by the Lamé potentials* $\varphi(x, t)$ *and* $\boldsymbol{\psi}(x, t)$ *defined through*

$$\begin{aligned} \varphi &= \Phi - 2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\cdot\boldsymbol{\Psi} - (1-2\nu)\,\boldsymbol{x}\cdot\boldsymbol{\Psi}, \\ \boldsymbol{\psi} &= 2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\times\boldsymbol{\Psi} + \nabla\times\boldsymbol{\Theta}, \end{aligned} \tag{38}$$

[8] This requirement is merely necessary for φ and $\boldsymbol{\psi}$ to be independent of t in the equilibrium case; conditions which are both necessary and sufficient will be given later. See the statement immediately following (57).

[9] See also [*22*] for a discussion of certain related questions.

provided $\boldsymbol{\Theta}(x)$ *is a particular solution of the pair of equations*

$$\nabla^2 \boldsymbol{\Theta} = 4(1-\nu)\boldsymbol{\Psi}, \quad \nabla\cdot\boldsymbol{\Theta} = 2(1-\nu)\,\boldsymbol{x}\cdot\boldsymbol{\Psi}. \tag{39}$$ [10]

To confirm the truth of this theorem, one needs merely substitute (38) into (7), (8), taking account of (4), (14), (36), (37), (39), and making use of the identities

$$\nu = \frac{c_1^2 - 2c_2^2}{2(c_1^2 - c_2^2)}, \tag{40}$$

$$\nabla^2(\boldsymbol{x}\cdot\boldsymbol{\Psi}) = \boldsymbol{x}\cdot\nabla^2\boldsymbol{\Psi} + 2\nabla\cdot\boldsymbol{\Psi}. \tag{41}$$

With a view toward motivating the particular form of (38), (39), we observe that if the potentials $\varphi(x,t)$, $\boldsymbol{\psi}(x,t)$ and $\Phi(x)$, $\boldsymbol{\Psi}(x)$ give rise to the same solution $\boldsymbol{u}(x)$ of (1), they must, according to (7) and (36), satisfy the relation

$$\nabla\varphi + \nabla\boldsymbol{\psi} = \nabla(\Phi + \boldsymbol{x}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\boldsymbol{\Psi}. \tag{42}$$

Also, in view of CLEBSCH's theorem, we may assume that (10) holds, *i.e.* $\boldsymbol{\psi}$ may be restricted to be solenoidal. Next, operate on (42) with the divergence and the curl, respectively, and recall (8), (10), (37), as well as the general vector identities (14) and (41). Thus,

$$\frac{\partial^2\varphi}{\partial t^2} = -4(1-\nu)\frac{\mu}{\varrho}\nabla\cdot\boldsymbol{\Psi}, \qquad \frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = 4(1-\nu)\frac{\mu}{\varrho}\nabla\times\boldsymbol{\Psi}, \tag{43}$$

and after two successive integrations,

$$\begin{aligned} \varphi &= -2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\cdot\boldsymbol{\Psi} + t\,\varphi_1(x) + \varphi_2(x), \\ \boldsymbol{\psi} &= 2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\times\boldsymbol{\Psi} + t\,\boldsymbol{\psi}_1(x) + \boldsymbol{\psi}_2(x). \end{aligned} \tag{44}$$

From (44), (10) follow

$$\nabla\cdot\boldsymbol{\psi}_1 = 0, \quad \nabla\cdot\boldsymbol{\psi}_2 = 0. \tag{45}$$

Now operate on the first and second of (44) with $\Box_1^2$ and $\Box_2^2$, respectively, bearing in mind (4), (8), and (37). This yields

$$\begin{aligned} 2(1-2\nu)\nabla\cdot\boldsymbol{\Psi} + t\nabla^2\varphi_1 + \nabla^2\varphi_2 &= 0, \\ -4(1-\nu)\nabla\times\boldsymbol{\Psi} + t\nabla^2\boldsymbol{\psi}_1 + \nabla^2\boldsymbol{\psi}_2 &= 0. \end{aligned} \tag{46}$$

Since (46) must hold identically with respect to t, we have

$$\nabla^2\varphi_1 = 0, \quad \nabla^2\varphi_2 = -2(1-2\nu)\nabla\cdot\boldsymbol{\Psi}, \tag{47}$$

$$\nabla^2\boldsymbol{\psi}_1 = 0, \quad \nabla^2\boldsymbol{\psi}_2 = 4(1-\nu)\nabla\times\boldsymbol{\Psi}. \tag{48}$$

By virtue of (41) and the second of (37), the general solution of the second of (47) may be written as

$$\varphi_2 = -(1-2\nu)\,\boldsymbol{x}\cdot\boldsymbol{\Psi} + \varphi_2', \quad \nabla^2\varphi_2' = 0. \tag{49}$$

On the other hand, the second of (45) and the second of (48) together assure the existence of functions $\boldsymbol{\Theta}(x)$ and $\lambda(x)$ such that

$$\boldsymbol{\psi}_2 = \nabla\times\boldsymbol{\Theta}, \quad \nabla^2\boldsymbol{\Theta} = 4(1-\nu)\boldsymbol{\Psi} + \nabla\lambda. \tag{50}$$

[10] As will be shown later, (39) always admit a simultaneous solution.

Substitution from (49), (50) into (44) leads to

$$\begin{aligned} \varphi &= -2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\cdot\boldsymbol{\Psi} - (1-2\nu)\,\boldsymbol{x}\cdot\boldsymbol{\Psi} + t\,\varphi_1 + \varphi_2', \\ \boldsymbol{\psi} &= 2(1-\nu)\frac{\mu t^2}{\varrho}\nabla\times\boldsymbol{\Psi} + t\boldsymbol{\psi}_1 + \nabla\times\boldsymbol{\Theta}, \end{aligned} \tag{51}$$

in which, because of (45), (47), (48), (49), and (50),

$$\nabla^2\varphi_1 = \nabla^2\varphi_2' = \nabla^2\boldsymbol{\psi}_1 = \nabla\cdot\boldsymbol{\psi}_1 = 0, \qquad \nabla^2\boldsymbol{\Theta} = 4(1-\nu)\boldsymbol{\Psi} + \nabla\lambda, \tag{52}$$

and λ is as yet arbitrary.

Finally, substitute (51) into (42), use (14), (37), (52), and equate the coefficients of like powers of t on either side of the resulting identity. In this manner one arrives at

$$\begin{gathered} \nabla\varphi_1 + \nabla\times\boldsymbol{\psi}_1 = 0, \\ \nabla(\varphi_2' + \nabla\cdot\boldsymbol{\Theta} - \lambda) = \nabla[\Phi + 2(1-\nu)\,\boldsymbol{x}\cdot\boldsymbol{\Psi}]. \end{gathered} \tag{53}$$

Equations (52), (53) can evidently be met by setting

$$\varphi_1 = \boldsymbol{\psi}_1 = \lambda = 0, \qquad \varphi_2' = \Phi, \tag{54}$$

if $\boldsymbol{\Theta}$ is a solution of (39), and (51) then become identical with (38). This accounts for the specific form of (38), (39).

The pair of partial differential equations (39) for $\boldsymbol{\Theta}(x)$ is compatible since, according to (41) and the second of (37),

$$\nabla^2(\boldsymbol{x}\cdot\boldsymbol{\Psi}) = 2\nabla\cdot\boldsymbol{\Psi}. \tag{55}$$

For given $\boldsymbol{\Psi}(x)$, a solution to (39) may be constructed by observing on the basis of the second of (37) and the second of (39) that $\boldsymbol{\Theta}(x)$ is biharmonic. Hence, in view of ALMANSI's [*23*] theorem, $\boldsymbol{\Theta}$ may be written as

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}' + x_1\boldsymbol{\Theta}'', \qquad \nabla^2\boldsymbol{\Theta}' = \nabla^2\boldsymbol{\Theta}'' = 0, \tag{56}$$

in which x_1 may alternatively be replaced with x_2, x_3, or $\boldsymbol{x}^2$. The two harmonic functions $\boldsymbol{\Theta}'(x)$ and $\boldsymbol{\Theta}''(x)$, in turn, may easily be determined in such a way that both of (39) hold.

The Lamé potentials (38) are seen, in general, to depend quadratically on the time. By (36), (37), and (41),

$$\nabla\cdot\boldsymbol{u} = -2(1-2\nu)\nabla\cdot\boldsymbol{\Psi}, \qquad \nabla\times\boldsymbol{u} = -4(1-\nu)\nabla\times\boldsymbol{\Psi}. \tag{57}$$

Consequently, φ and $\boldsymbol{\psi}$ in (38) *are independent of the time in the equilibrium case if and only if the displacement field is both equivoluminal and irrotational*, which, by virtue of (1), requires that $\boldsymbol{u}(x)$ be harmonic. As is apparent from the theorem at the beginning of this section, *every equilibrium solution $\boldsymbol{u}(x)$ of* (1) *may be represented as a sum of an irrotational and an equivoluminal motion the time-dependence of which cancels upon addition.* This decomposition is evidently not unique.

To illustrate the foregoing conclusions we turn to a specific example. Thus, consider KELVIN's elastostatic solution [11] appropriate to a concentrated load at

[11] See LOVE [*25*], art. 130.

a point of a medium occupying the entire space. Let the load have unit magnitude and be applied at the origin in the direction of the x_α-axis ($\alpha = 1, 2, 3$). The corresponding PAPKOVICH-NEUBER potentials then take the indicial form

$$\Phi = 0, \qquad \Psi_i = -\frac{\delta_{\alpha i}}{16\pi(1-\nu)\mu R}, \tag{58}$$

where R is the distance from the origin, while $\delta_{\alpha i}$ denotes the Kronecker delta. In accordance with (36), the equilibrium displacement field generated by the harmonic stress functions (58) has the components

$$u_i = \frac{1}{16\pi(1-\nu)\mu R}\left[\frac{x_\alpha x_i}{R^2} + (3-4\nu)\,\delta_{\alpha i}\right]. \tag{59}$$

In this instance one finds that (39) may be met by taking

$$\Theta_i = -\frac{\delta_{\alpha i} R}{8\pi\mu}, \tag{60}$$

so that (38) yield the associated Lamé potentials

$$\begin{aligned} \varphi &= -\frac{x_\alpha}{16\pi(1-\nu)\mu R}\left[\frac{2(1-\nu)\mu t^2}{\varrho R^2} - 1 + 2\nu\right], \\ \psi_i &= \frac{\varepsilon_{\alpha i j} x_j}{8\pi\mu R}\left[\frac{\mu t^2}{\varrho R^2} - 1\right], \end{aligned} \tag{61}$$

in which $\varepsilon_{\alpha i j}$ are the components of the usual alternator and summation over repeated indices is implied. The stress functions (61) conform to the wave equations (8). Using (61) in conjunction with (7), we arrive at the decomposition

$$u_i(x) = u_i'(x,t) + u_i''(x,t), \tag{62}$$

where

$$\begin{aligned} u_i' &= \frac{\partial\varphi_i}{\partial x_i} = \frac{1}{16\pi(1-\nu)\mu R}\left[\frac{2(1-\nu)\mu t^2}{\varrho R^2}\left(\frac{3x_\alpha x_i}{R^2} - \delta_{\alpha i}\right) - (1-2\nu)\left(\frac{x_\alpha x_i}{R^2} - \delta_{\alpha i}\right)\right], \\ u_i'' &= \varepsilon_{ikj}\frac{\partial\psi_j}{\partial x_k} = \frac{1}{8\pi\mu R}\left[\frac{\mu t^2}{\varrho R^2}\left(\delta_{\alpha i} - \frac{3x_\alpha x_i}{R^2}\right) + \delta_{\alpha i} + \frac{x_\alpha x_i}{R^2}\right]. \end{aligned} \tag{63}$$

The irrotational and the equivoluminal motions represented by $u_i'(x,t)$ and $u_i''(x,t)$, respectively, individually satisfy the equations of motion (1).

The resolution of an equilibrium solution into dynamic component solutions has an analogue in the theory of the biharmonic equation and of the repeated wave equation. To make this clear we recall a result established separately in [*19*] and contained in a more general theorem due to BOGGIO [*24*]: every (sufficiently regular) solution $g(x,t)$ of the equation

$$\Box_1^2 \Box_2^2 g = 0 \tag{64}$$

admits the representation

$$g(x,t) = g' + g'', \tag{65}$$

where the functions $g'(x,t)$ and $g''(x,t)$ satisfy the ordinary wave equations

$$\Box_1^2 g' = 0, \qquad \Box_2^2 g'' = 0. \tag{66}$$

The operators $\Box_1^2$, $\Box_2^2$ appearing here are again given by the first of (4), but c_1 and c_2 at present may have any distinct non-zero values.

If, in particular, $\boldsymbol{g}$ happens to be independent of t, (64) degenerates into the biharmonic equation $\nabla^4 \boldsymbol{g}=0$, the complete integral of which admits ALMANSI'S [*23*] representation in terms of two harmonic functions. Thus,

$$\boldsymbol{g}(x)=\boldsymbol{h}'+\boldsymbol{x}\cdot\boldsymbol{h}'', \tag{67}$$

in which $\boldsymbol{h}'(x)$ and $\boldsymbol{h}''(x)$ satisfy

$$\nabla^2\boldsymbol{h}'=0, \quad \nabla^2\boldsymbol{h}''=0. \tag{68}$$

The representations (65), (66) and (67), (68) may be regarded as respective counterparts of LAMÉ'S general solution (7), (8) to the equations of motion and of the Papkovich-Neuber solution (36), (37) to the equilibrium equations. BOGGIO'S theorem, and hence (65), (66), still hold if $\boldsymbol{g}$ is a function of position alone, but in this event the resolving functions $\boldsymbol{g}'$ and $\boldsymbol{g}''$ ordinarily remain time-dependent. For instance, the scalar biharmonic function

$$g(x)=x_1^2 \tag{69}$$

is the sum of the two wave functions

$$g'=\frac{c_2^2(x_1^2+c_1^2t^2)}{c_2^2-c_1^2}, \quad g''=-\frac{c_1^2(x_1^2+c_2^2t^2)}{c_2^2-c_1^2}. \tag{70}$$

4. Connection between Lamé's and Iacovache's Solution

IACOVACHE'S [*16*] solution of the equations of motion (1) is given by

$$\boldsymbol{u}(x,t)=2(1-\nu)\,\Box_1^2\boldsymbol{G}-\nabla\nabla\cdot\boldsymbol{G}, \tag{71}$$

where

$$\Box_1^2\,\Box_2^2\,\boldsymbol{G}=0. \tag{72}$$

Our present aim is to examine the relation between (71), (72) and LAMÉ'S solution (34), (35).

Invoking once again the special case of BOGGIO'S [*24*] theorem arrived at independently in [*19*], we note first that there exist functions $\boldsymbol{G}'(x,t)$, $\boldsymbol{G}''(x,t)$ such that

$$\boldsymbol{G}(x,t)=\boldsymbol{G}'+\boldsymbol{G}'' \tag{73}$$

and

$$\Box_1^2\boldsymbol{G}'=0, \quad \Box_2^2\boldsymbol{G}''=0. \tag{74}$$

From (4) and the second of (74) follows

$$2(1-\nu)\,\Box_1^2\boldsymbol{G}''=\nabla^2\boldsymbol{G}''. \tag{75}$$

Using (73), (75), we may write (71) as

$$\boldsymbol{u}=\nabla^2\boldsymbol{G}''-\nabla\nabla\cdot(\boldsymbol{G}'+\boldsymbol{G}'') \tag{76}$$

from which, on applying the general identity (14) to the vector field $\boldsymbol{G}''(x,t)$, we draw

$$\boldsymbol{u}=-\nabla\nabla\cdot\boldsymbol{G}'-\nabla\times\nabla\times\boldsymbol{G}''. \tag{77}$$

If, finally, we define functions $\varphi(x,t)$ and $\boldsymbol{\psi}(x,t)$ by

$$\varphi=-\nabla\cdot\boldsymbol{G}', \quad \boldsymbol{\psi}=-\nabla\times\boldsymbol{G}'', \tag{78}$$

(71) takes the form

$$\boldsymbol{u}(x,t) = \nabla\varphi + \nabla\times\boldsymbol{\psi}, \quad \nabla\cdot\boldsymbol{\psi} = 0, \tag{79}$$

while (74), (78) yield

$$\Box_1^2\varphi = 0, \quad \Box_2^2\boldsymbol{\psi} = 0. \tag{80}$$

Equations (79), (80), however, are identical with (34), (35). IACOVACHE's *solution thus reduces to* LAMÉ's *solution of* (1) *if the stress function* $\boldsymbol{G}(x, t)$ *is subjected to the transformation* (73), (74), (78). In view of the completeness of IACOVACHE's solution (71), (72), established previously in [*19*], the preceding reduction supplies an alternative proof for the completeness of LAMÉ's solution (34), (35). In comparing this proof with DUHEM's [*9*] argument, given in Section 2, it should be remembered that we have at present assumed the generality of IACOVACHE's solution which was proved in [*19*] with the aid of the theory of retarded potentials.

If $\boldsymbol{G}$ is independent of the time, (71), (72) degenerate into GALERKIN's [*18*] general equilibrium solution

$$\boldsymbol{u}(x) = 2(1-\nu)\nabla^2\boldsymbol{G} - \nabla\nabla\cdot\boldsymbol{G}, \tag{81}$$

$$\nabla^4\boldsymbol{G} = 0, \tag{82}$$

of which IACOVACHE's solution is therefore a dynamic generalization. GALERKIN's solution (81), (82), in turn, as pointed out by MINDLIN [*21*], is reducible to the PAPKOVICH-NEUBER solution (36), (37) by setting

$$\Phi(x) = -\nabla\cdot\boldsymbol{G} - \boldsymbol{x}\cdot\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}(x) = -\tfrac{1}{2}\nabla^2\boldsymbol{G}. \tag{83}$$

Similarly, as observed in [*19*], the time-dependent transformation

$$\Phi(x,t) = -\nabla\cdot\boldsymbol{G} - \boldsymbol{x}\cdot\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}(x,t) = -\tfrac{1}{2}\Box_1^2\boldsymbol{G}, \tag{84}$$

applied to $\boldsymbol{G}(x, t)$ of (71), (72), leads to

$$\boldsymbol{u}(x,t) = \nabla(\Phi + \boldsymbol{x}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\boldsymbol{\Psi}, \tag{85}$$

$$\Box_1^2\Phi = -\boldsymbol{x}\cdot\Box_1^2\boldsymbol{\Psi}, \quad \Box_2^2\boldsymbol{\Psi} = 0, \tag{86}$$

which may be regarded as a dynamic generalization of (36), (37). Since the potentials Φ and $\boldsymbol{\Psi}$ in (86) are coupled, this form of the complete solution to the equations of motion (1) is of no practical interest.

In attempting to judge the comparative merits of IACOVACHE's solution (71), (72) and of LAMÉ's solution (79), (80), the following considerations would appear to be pertinent. IACOVACHE's solution remains complete in the equilibrium case if the generating potential $\boldsymbol{G}(x, t)$ is taken to be a function of position alone. In contrast, as explained earlier, LAMÉ's solution has no equilibrium counterpart in the foregoing sense. This circumstance reflects the relative economy of LAMÉ's solution to the equations of motion, which is simpler in structure than the available complete solutions to the equations of equilibrium. It is clear from (73), (74) that the single stress function $\boldsymbol{G}$ in (71), which satisfies the repeated wave equation (72), is equivalent to two stress functions each of which meets an ordinary wave equation. The number of independent solutions of the wave equation involved in (71), (72) and (79), (80), respectively, is thus the same. On the other hand,

(71) contains space derivatives of $\boldsymbol{G}$ up to the second order while only first-order derivatives of φ and $\boldsymbol{\psi}$ are seen to enter (79). Finally, (79), (80) are conveniently transformed into general orthogonal curvilinear coordinates, whereas (71), (72) give rise to exceedingly cumbersome forms when referred to curvilinear coordinates, with the exception of circular cylindrical coordinates. For all of these reasons, Lamé's solution deserves preference over Iacovache's general solution to the equations of motion in applications of elastokinetic theory.

Noll [20] recently dealt with the integration of the equations of motion in the case of steady-state oscillations of elastic bodies. He thus considered motions of the type

$$\boldsymbol{u}(x,t) = \boldsymbol{u}_0(x)\, e^{i\omega t} \tag{87}$$[12]

and showed that if $\boldsymbol{u}(x,t)$ given by (87) satisfies (1), the amplitude function $\boldsymbol{u}_0(x)$ admits the representation

$$\boldsymbol{u}_0(x) = 2(1-\nu)\,\nabla^2 \boldsymbol{G}_0 + (1-2\nu)\,\frac{\varrho\,\omega^2}{\mu}\,\boldsymbol{G}_0 - \nabla\nabla\cdot\boldsymbol{G}_0, \tag{88}$$[13]

$$\left[\nabla^2 + \frac{\omega^2}{c_1^2}\right]\left[\nabla^2 + \frac{\omega^2}{c_2^2}\right]\boldsymbol{G}_0 = 0. \tag{89}$$

The solution (87), (88), (89) is identified as a special case of Iacovache's solution by taking

$$\boldsymbol{G}(x,t) = \boldsymbol{G}_0(x)\, e^{i\omega t} \tag{90}$$

in (71), (72). A more useful complete representation of the steady-state vibration (87), however, is obtained by setting

$$\varphi(x,t) = \varphi_0(x)\, e^{i\omega t}, \qquad \boldsymbol{\psi}(x,t) = \boldsymbol{\psi}_0(x)\, e^{i\omega t} \tag{91}$$

in Lamé's general solution (79), (80).

5. Axisymmetric and Two-dimensional Solutions of the Equations of Motion

Let (r,γ,z) be circular cylindrical coordinates defined by the mapping

$$\begin{gathered} x_1 = r\cos\gamma, \qquad x_2 = r\sin\gamma, \qquad x_3 = z, \\ 0 \leqq r < \infty, \qquad 0 \leqq \gamma < 2\pi, \qquad -\infty < z < \infty, \end{gathered} \tag{92}$$

and let (u_r, u_γ, u_z) stand for the corresponding cylindrical components of the displacement vector. A displacement field possesses (torsionless) rotational symmetry about the z-axis if

$$u_r = u_r(r,z,t), \qquad u_z = u_z(r,z,t), \qquad u_\gamma = 0. \tag{93}$$

Suppose we denote the cylindrical components of the vector potential $\boldsymbol{\psi}$ in Lamé's solution by $(\psi_r, \psi_\gamma, \psi_z)$. Evidently, (79), (80) represent a solution of (1) of the form (93) provided we choose

$$\varphi = \varphi(r,z,t), \qquad \psi_\gamma = \psi(r,z,t), \qquad \psi_r = \psi_z = 0, \tag{94}$$

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[12] The fact that ω here designates the circular frequency while $\boldsymbol{\omega}$ in (5) denotes the rotation vector ought not to cause confusion.

[13] $2(1-\nu)\,\boldsymbol{G}_0 = \boldsymbol{q}(x)$ in the notation of [20].

and the associated non-vanishing cylindrical displacement components then become

$$u_r = \frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z}, \qquad u_z = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r}. \tag{95}$$

The two scalar potentials φ, ψ satisfy the wave equations

$$\Box_1^2 \varphi = 0, \qquad \Box_2^2 \psi = 0, \tag{96}$$

in which $\Box_1^2$, $\Box_2^2$ retain their previous definition (4), and the Laplacian operator now appears as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{97}$$

Equations (95), (96) constitute the axisymmetric form of LAMÉ's solution.

A rotationally symmetric version of IACOVACHE's solution is obtained from (71), (72) by setting

$$G_3 = G(r, z, t), \qquad G_1 = G_2 = 0 \tag{98}$$

where (G_1, G_2, G_3) are the cartesian components of $\boldsymbol{G}$. Consequently,

$$u_r = -\frac{\partial^2 G}{\partial r\,\partial z}, \qquad u_z = 2(1-\nu)\nabla^2 G - \frac{\partial^2 G}{\partial z^2} - (1-2\nu)\frac{\varrho}{\mu}\frac{\partial^2 G}{\partial t^2}, \tag{99}$$

$$\Box_1^2 \Box_2^2 G = 0, \tag{100}$$

and $\Box_1^2$, $\Box_2^2$ have the same meaning as in (96). If, in particular, G here is independent of the time, the foregoing solution coincides (except for a constant factor) with LOVE's [25][14] general solution of the equations of equilibrium in the presence of rotational symmetry.

The connection between the alternative axisymmetric solutions (95), (96) and (99), (100) follows at once from an appropriate specialization of the general conclusions reached in the preceding section of this paper. Thus, (73), (74), (78), because of (94), (98), yield in the present instance

$$G(r, z, t) = G' + G'', \tag{101}$$

$$\Box_1^2 G' = 0, \qquad \Box_2^2 G'' = 0, \tag{102}$$

$$\varphi = \frac{\partial G'}{\partial z}, \qquad \psi = \frac{\partial G''}{\partial r}. \tag{103}$$

The transformation (101), (102), (103) reduces (99), (100) to (95), (96), as was observed independently by PREDELEANU [26]. Equations (95), (96) and (99), (100) individually represent complete solutions of the equations of motion (1) in the rotationally symmetric case governed by (93). This is readily shown to be true of IACOVACHE's solution (99), (100) on the basis of a suitable adaptation of the general completeness proof presented in [19]; the completeness of LAMÉ's solution (95), (96) may then be inferred directly with the aid of (101), (102), (103).

We turn, finally, to a discussion of general solutions of the equations of motion in the two-dimensional theory of elasticity. In the event of plane strain (parallel to the plane $x_3 = 0$) the displacement field is characterized by

$$u_1 = u_1(x_1, x_2, t), \qquad u_2 = u_2(x_1, x_2, t), \qquad u_3 = 0. \tag{104}$$

[14] See art. 188, p. 274.

It is clear from the two-dimensional counterpart of CLEBSCH's theorem (Section 2) that LAMÉ's solution remains complete in the present instance if the generating potentials are taken in the restricted form

$$\varphi = \varphi(x_1, x_2, t), \quad \psi_3 = \psi(x_1, x_2, t), \quad \psi_1 = \psi_2 = 0. \tag{105}$$

Equations (79), (80) now lead to

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}, \quad u_2 = \frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}, \tag{106}$$

$$\Box_1^2 \varphi = 0, \quad \Box_2^2 \psi = 0. \tag{107}$$

A plane-strain version of IACOVACHE's solution, on the other hand, is obtained by setting

$$G_2 = G(x_1, x_2, t), \quad G_1 = G_3 = 0 \tag{108}$$

in (71), (72). This choice, after some elementary manipulations involving (4), results in

$$u_1 = -\frac{\partial^2 G}{\partial x_1 \, \partial x_2}, \quad u_2 = 2(1-\nu)\frac{\partial^2 G}{\partial x_1^2} + (1-2\nu)\frac{\partial^2 G}{\partial x_2^2} - \frac{(1-2\nu)\varrho}{\mu}\frac{\partial^2 G}{\partial t^2}, \tag{109}$$

$$\Box_1^2 \Box_2^2 G = 0. \tag{110}$$

Equations (109), (110) are identical (except for a constant factor) with the complete two-dimensional solution deduced in a different manner by SOBRERO [27]. SOBRERO's general solution is reducible to the two-dimensional form (106), (107) of LAMÉ's solution by employing once again the transformation (73), (74), (78). Bearing in mind (105), (108), we thus arrive at

$$G(x_1, x_2, t) = G' + G'', \tag{111}$$

$$\Box_1^2 G' = 0, \quad \Box_2^2 G'' = 0, \tag{112}$$

$$\varphi = -\frac{\partial G'}{\partial x_2}, \quad \psi = -\frac{\partial G''}{\partial x_1}, \tag{113}$$

which link the two solutions (106), (107) and (109), (110). A plane-strain solution of (1) in terms of *two* scalar potentials each satisfying the iterated wave equation (110), was given by G. C. MOISIL [28][15]. MOISIL's result is also found to be a special case of IACOVACHE's solution (71), (72) and is reached (except for a constant factor) by choosing

$$G_1 = G_1(x_1, x_2, t), \quad G_2 = G_2(x_1, x_2, t), \quad G_3 = 0. \tag{114}$$

The present discussion has so far been confined to plane strain. As is well known, the transition to the corresponding results for generalized plane stress is effected by replacing POISSON's ratio ν with $\nu/(1+\nu)$. While the wave velocity c_2 in (4) remains invariant under this change of POISSON's ratio, c_1^2 passes over into $\hat{c}_1^2$, where

$$\hat{c}_1^2 = \frac{2\mu}{(1-\nu)\varrho}. \tag{115}$$

[15] See Equations (116) on p. 384 of [28]. The coefficient $\lambda + 2\mu$ appearing in both of these equations is in error and should read $\lambda + \mu$.

Equations (106), (107) have been employed in a number of recent two-dimensional studies dealing with uniformly moving loads, punches, cracks, and dislocations[16] as, for example, by SNEDDON [*30*]. In contrast, RADOK [*31*] reconsidered several plane problems of motion on the basis of a dynamic analogue of AIRY's elastostatic stress function. The same scheme was encountered later on by TEODORESCU [*32*].

The comments made at the end of Section 4 concerning the advantage of LAMÉ's over IACOVACHE's form of the general solution to the three-dimensional equations of motion, appear to be equally valid in connection with axisymmetric and plane problems. In fact, it is probably safe to say that the various displacement potentials (or stress functions) which have been introduced into elastokinetics as a result of efforts to generalize known stress functions of elastostatics, while interesting from the theoretical viewpoint, are of limited interest from the point of view of applications. For the requirement that a complete solution of the equations of motion reduce to a complete solution of the equilibrium equations whenever the generating stress functions are independent of the time precludes the utilization of an element of relative simplicity which is inherent in the dynamic case but absent from the degenerate equilibrium case. This fact is reflected clearly in the comparative economy of LAMÉ's classical general integral of the equations of motion which has no such counterpart in the equilibrium theory.

Note Added in Proof. The author recently encountered certain publications of which he was previously unaware and which necessitate the following additional comments:

(1) Correct proofs of CLEBSCH's [*7*] completeness theorem were also given independently by SOMIGLIANA [*34*][17] (1892) and by TEDONE [*35*] (1897) prior to the appearance of DUHEM's [*9*] proof in 1898. Also, it is evident from the introductory remarks in [*34*] that SOMIGLIANA recognized the shortcomings of CLEBSCH's earlier argument[18]. Nevertheless, DUHEM's proof, which is summarized in Section 2 of the present paper, appears to be more economical and direct than the alternative proofs contained in [*34*], [*35*].

(2) Further, SOMIGLIANA [*34*] (1892), deduces the representation[19] (71), (72), from LAMÉ's solution (7), (8) in a manner which assures the completeness of (71), (72). Consequently, the general solution (71), (72), which here and throughout the contemporary literature has been attributed to IACOVACHE [*16*] (1949), should instead be credited to SOMIGLIANA.

(3) Finally, reference should be made to recent papers by ARZHANIKH [*36*] and by SOLOMON [*37*], which contain variants of the solution (85), (86) in terms of stress functions satisfying a pair of interlocked partial differential equations of the second order.

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[16] See [*11*] and [*29*] for references to some of these investigations.

[17] See the additional references at the end of this note.

[18] "Ora il procedimento seguito ... per dimostrare il teorema in discorso ... richiede, a mio credere, qualche altra considerazione che lo completi".

[19] See Equations (16), (18) in [*34*].

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Korn Inequalities for the Sphere and Circle

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1. Introduction *

Let $\mathfrak{G}$ be a given region in either two or three dimensions. Let $\mathscr{C}^n$ denote the class of all vector fields $\boldsymbol{u}(\boldsymbol{x})$, continuous in the closure $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ and n times continuously differentiable in $\mathfrak{G}$ such that the Dirichlet integral

$$D(\boldsymbol{u}) = \int_{\mathfrak{G}} u_{i,j}\, u_{i,j}\, dv \tag{1.1}$$

is finite. Let d_{ij} and ω_{ij} denote the symmetric and antisymmetric parts of the gradient of $\boldsymbol{u}$:

$$\begin{aligned} d_{ij} &= u_{(i,j)} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ \omega_{ij} &= u_{[i,j]} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \end{aligned} \tag{1.2}$$

We shall also use the abbreviated notation

$$d^2 = d_{ij}\, d_{ij}, \qquad \omega^2 = \omega_{ij}\, \omega_{ij}, \qquad \operatorname{tr} d = d_{kk}. \tag{1.3}$$

The summation convention, Cartesian coordinates, and Cartesian tensor notation will be used throughout.

A *Korn inequality* states the existence of a number K which depends only on the shape of the region $\mathfrak{G}$ such that

$$\int_{\mathfrak{G}} \omega^2\, dv \leqq K \int_{\mathfrak{G}} d^2\, dv \tag{1.4}$$

for all vector fields which satisfy one or more side conditions. That an inequality of the form (1.4) cannot hold for arbitrary vector fields in $\mathscr{C}^\infty$ follows from the existence of pure rotations, $u_i = e_{ijk}\, x_j\, h_k$, $\boldsymbol{h}$ = constant vector, $e_{123} = 1$, $e_{ijk} = e_{[ijk]}$.

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* While this paper was being written, the authors were in touch with L. E. Payne & H. F. Weinberger, who, by using a variational approach, were able to obtain extrema for the expression (1.6) in the second case for the sphere and the circle. But they were not at that time able to prove that their largest extremum was actually the supremum of (1.6) and hence Korn's constant in the second case. However, after this paper had gone to press, the authors were delighted to learn that Payne & Weinberger had succeeded in formulating a proof of this, and thus Korn's constant in the second case is now known to be 43/13 for the sphere and 3 for the circle. Payne & Weinberger will publish their work in a subsequent issue of this Archive.

The authors were dissuaded from withdrawing this paper in the light of the results of Payne & Weinberger because, although the numerical values of upper bounds for Korn's constant contained herein are no longer of value, the remainder of the work remains a contribution. Also the adaptation of the method of Friedrichs contained herein is applicable to more general regions, whereas Payne & Weinberger presently feel that their method is not.

We define, with Friedrichs [1], $\mathscr{D}^\bullet$ to be that subset of $\mathscr{C}^1$ consisting of those fields $^\bullet\boldsymbol{u}$ which vanish outside some compact subset of $\mathfrak{G}$, and $\mathscr{D}^\circ$ to be the set of fields $\boldsymbol{u}$ in $\mathscr{C}^1$ for which fields $^\bullet\boldsymbol{u}$ in $\mathscr{D}^\bullet$ may be found to make $D(\boldsymbol{u} - {}^\bullet\boldsymbol{u})$ arbitrarily small.

Korn inequalities for a class of regions, called *Ω-domains* by Friedrichs [1], are known to exist. For a definition of the class of Ω-domains the reader is referred to [1]. It suffices here merely to note that this class of regions includes the sphere and the circle. Moreover, for an Ω-domain, Korn inequalities exist under three types of side conditions. Following Korn and Friedrichs, we consider separately the three sets of side conditions which, using the terminology of Friedrichs, we call the *three cases:*

$$\begin{aligned} &\textit{First case:} && \boldsymbol{u} \in \mathscr{D}^\circ, \\ &\textit{Second case:} && \int_{\mathfrak{G}} \omega_{ij}\, dv = 0, \quad \boldsymbol{u} \in \mathscr{C}^1, \\ &\textit{Main Case:} && \int \omega_{ij}\, dv = 0, \quad \boldsymbol{u} \in \mathscr{C}^3, \quad \textit{and} \quad d_{ij,j} = 0. \end{aligned} \tag{1.5}$$

If a Korn inequality exists for a given region $\mathfrak{G}$, subject to a given one of the side conditions (1.5), then the set of real numbers which are the values of

$$\frac{\int_{\mathfrak{G}} \omega^2\, dv}{\int_{\mathfrak{G}} d^2\, dv} \tag{1.6}$$

subject to the given side conditions, has an upper bound, and hence a least upper bound. We call this least upper bound *Korn's constant for the region $\mathfrak{G}$ in the first case, the second case, and the main case* respectively. We denote these dimensionless positive numbers by $K^1_{\mathfrak{G}}$, $K^2_{\mathfrak{G}}$, and $K^M_{\mathfrak{G}}$. Any number greater than or equal to $K^\Omega_{\mathfrak{G}}$, $\Omega = 1, 2, M$, will be called an *upper bound for $K^\Omega_{\mathfrak{G}}$*, and any number smaller than or equal to $K^\Omega_{\mathfrak{G}}$ will be called a *lower bound* for $K^\Omega_{\mathfrak{G}}$. Substitution of any upper bound for a Korn constant in (1.4) yields an inequality valid for all vector fields satisfying the appropriate set of side conditions (1.5).

The proof of the existence of a Korn inequality in the first case for an Ω-domain follows immediately from the identity

$$\int_{\mathfrak{G}} \omega^2\, dv = \int_{\mathfrak{G}} d^2\, dv - \int_{\mathfrak{G}} (\operatorname{tr} d)^2\, dv, \tag{1.7}$$

which is shown by Friedrichs [1] to hold for all $\boldsymbol{u}$ in $\mathscr{D}^\circ$ when $\mathfrak{G}$ is an Ω-domain. Clearly, then, for the first case

$$\int_{\mathfrak{G}} \omega^2\, dv \leqq \int_{\mathfrak{G}} d^2\, dv, \tag{1.8}$$

and $K^1_{\mathfrak{G}} \leqq 1$ for all $\mathfrak{G}$. Moreover, since it is easy to construct vector fields $\boldsymbol{u} \in \mathscr{D}^\bullet$ for which $\operatorname{tr} d = 0$, we see from (1.7) that $K^1_{\mathfrak{G}} = 1$ for any Ω-domain.

The useful role of the identity (1.7) and of Korn's inequality in the first case which follows from it in studies of the displacement boundary value problem of linear elasticity theory is made apparent in the papers of Korn [2, 3] and Friedrichs [1].

To prove the existence of a Korn inequality in the second and main cases is not so easy. As shown by FRIEDRICHS and in Section 4, the second case follows easily from the other two, whence the name, the "main case". We base our understanding of the main case on the paper by FRIEDRICHS. With him we confess inability to follow KORN'S original treatment.

The constructive proof in [1] of the existence of a Korn inequality in the second and main cases provides a technique for computing, at least in principle, upper bounds for the Korn constants $K_{\mathfrak{G}}^M$ and $K_{\mathfrak{G}}^2$ for any Ω-domain. But to our knowledge, no one has found a value for the Korn constants $K_{\mathfrak{G}}^2$ or $K_{\mathfrak{G}}^M$ for any region. Nor have we seen in the literature any upper bound for these numbers for any region whatsoever. Our curiosity was aroused as to just how large or how small the Korn constants might be for even a region so simple as a sphere or a circle. For these regions it turns out that FRIEDRICHS's construction for the second and main cases can be simplified somewhat, though we retain its essential ingredients. To the simplified proof for the sphere and circle we add a determination of the following upper and lower bounds:

$$\begin{aligned} &\text{For the sphere:} \quad 3 \leqq K_{\mathfrak{G}}^2 < 20, \quad 3 \leqq K_{\mathfrak{G}}^M < 17 \\ &\text{For the circle:} \quad \tfrac{4}{3} \leqq K_{\mathfrak{G}}^2 \leqq 6, \quad K_{\mathfrak{G}}^M \leqq 4. \end{aligned} \tag{1.9}$$

It is our belief that the method developed below for the sphere and the circle can be extended to more general regions.

2. Lower bounds for Korn's constants

a) The case of three dimensions. For the purposes of this section let $\mathfrak{G}$ be any region for which the centroid and moment of inertia tensor about the centroid (EULER'S tensor) have meaning. Let the origin of the Cartesian coordinate system x_i coincide with the centroid of $\mathfrak{G}$, so that the moment of inertia tensor about the centroid is given by

$$I_{ij} = \int_{\mathfrak{G}} x_i\, x_j\, dv. \tag{2.1}$$

Let $\boldsymbol{h}$ be a constant unit vector field, $h_i h_i = 1$, and consider the vector field

$$u_i = (h_k\, x_k)\, e_{ipq}\, h_p\, x_q. \tag{2.2}$$

Viewed as a displacement field of an elastic body say, the field $\boldsymbol{u}$ represents a uniform twisting in which each particle in a plane normal to $\boldsymbol{h}$ remains in that plane but is displaced in a direction normal to $\boldsymbol{h}$ and its position vector $\boldsymbol{x}$.

We obtain for the vector field (2.2)

$$\begin{aligned} &\int_{\mathfrak{G}} d^2\, dv = \tfrac{1}{2}(\delta_{ij} - h_i h_j)\, I_{ij}, \\ &\int_{\mathfrak{G}} \omega^2\, dv = \int_{\mathfrak{G}} d^2\, dv + 2 h_i h_j I_{ij}, \\ &\qquad d_{ij,j} = 0, \end{aligned}$$

from which it follows that

$$\frac{\int_{\mathfrak{G}} \omega^2\, dv}{\int_{\mathfrak{G}} d^2\, dv} = 1 + \frac{4 I_{ij} h_i h_j}{(\delta_{ij} - h_i h_j)\, I_{ij}}. \tag{2.3}$$

Also, since the derivatives of u_i are linear and homogeneous in x_k, and since the origin is at the centroid of $\mathfrak{G}$, $(1.5)_2$ is satisfied. Furthermore, since $d_{ij,j}=0$, our example belongs to the main case.

In general, the right-hand side of (2.3) will vary with the choice of the direction $\boldsymbol{h}$. As can be shown by the method of Lagrange multipliers, it will have extrema when $\boldsymbol{h}$ is in the direction of the principal axes of the moment of inertia tensor for the region. The largest we can make the right-hand side by varying the unit vector $\boldsymbol{h}$ is

$$1+\frac{4I_{\max}}{I_{\text{med}}+I_{\min}}, \tag{2.4}$$

where $I_{\max}\geqq I_{\text{med}}\geqq I_{\min}$ are the eigenvalues of the moment of inertia tensor.

Considering different regions now, (2.4) will have the smallest value for a region for which the three eigenvalues are all equal, in which case its value will be 3. Thus for any region in three dimensions $K_{\mathfrak{G}}^M\geqq 3$, $K_{\mathfrak{G}}^2\geqq 3$. Since (2.4) yields a lower bound for $K_{\mathfrak{G}}^M$ (and $K_{\mathfrak{G}}^2$) for any given region we may make the following observations:

1. There is no single number K which will yield a Korn inequality in the second or main cases for all three-dimensional regions—*i.e.*, the set of Korn constants in the second and main cases for three-dimensional regions has no upper bound.

2. Given any region $\mathfrak{G}$ in three dimensions, one can find a sufficiently long and thin right circular cylinder $\mathfrak{G}'$ whose Korn constant $K_{\mathfrak{G}'}^M$ exceeds $K_{\mathfrak{G}}^M$.

b) The case of two dimensions. Next we establish a lower bound for Korn's constant $K_{\mathfrak{G}}^2$ for regions in the plane. For this purpose we compute the ratio (1.7) for the 2-dimensional vector field $\boldsymbol{u}$ having the form

$$u_i=(h_k\,x_k)\,e_{ij}\,x_j. \tag{2.5}$$

All indices now take the values 1 and 2, and $e_{12}=1$, $e_{[ij]}=e_{ij}$. Again we assume the origin of the coordinate system coincides with the centroid of the planar region and let

$$J_{ij}=\int_{\mathfrak{G}} x_i\,x_j\,dv \tag{2.6}$$

denote the moment of inertia tensor of the planar region $\mathfrak{G}$ about its centroid. For a vector field having the form (2.5) we get

$$2\int_{\mathfrak{G}} d^2\,dv=2\operatorname{tr}J-J_{ij}\,h_i\,h_j,$$

$$2\int_{\mathfrak{G}}\omega^2\,dv=4\,J_{ij}\,h_i\,h_j,$$

from which it follows that

$$\frac{\int_{\mathfrak{G}}\omega^2\,dv}{\int_{\mathfrak{G}} d^2\,dv}=\frac{4\,J_{ij}\,h_i\,h_j}{2\operatorname{tr}J-J_{ij}\,h_i\,h_j}. \tag{2.7}$$

The maximum value of the right-hand side of (2.7) is attained when $\boldsymbol{h}$ is an eigenvector of J_{ij} corresponding to its maximum eigenvalue. Thus the maximum value for the ratio in (2.7) is

$$\frac{4\,J_{\max}}{J_{\max}+2\,J_{\min}}, \tag{2.8}$$

which lies between $\frac{4}{3}$ and 4. It follows that for any region in the plane, $K_{\mathfrak{G}}^2\geqq\frac{4}{3}$. The lower bounds announced in Section 1 are established.

3. Some differential identities and algebraic inequalities

In what follows we shall make repeated use of the identities

$$\omega_{jk,l} = d_{lj,k} - d_{kl,j}, \tag{3.1}$$

$$d_{ij,kl} + d_{kl,ij} - d_{ik,jl} - d_{jl,ik} = 0, \tag{3.2}$$

and of the inequality,

$$(\operatorname{tr} d)^2 \leqq n\, d^2, \tag{3.3}$$

where n is the dimension of the space. The differential identities (3.1) and (3.2) are well known in elasticity theory. The latter are the familiar *compatibility conditions* that d_{ij} be expressible in the form $(1.2)_1$. Inequality (3.3) follows easily from CAUCHY's inequality, $2ab \leqq a^2 + b^2$, in a frame of reference where d_{ij} is diagonal. Since both members of the inequality are invariant under orthogonal transformations, (3.3) is valid always.

From (3.1) we get

$$\omega_{ij,k}\,\omega_{ij,k} = 2\left(d_{ij,k}\, d_{ij,k} - d_{ij,k}\, d_{ki,j}\right). \tag{3.4}$$

We shall be concerned in particular with vector fields which satisfy the differential equations $d_{ij,j} = 0$ of the main case. For this class of vector fields we get from (3.1) and (3.2):

$$\begin{aligned} \nabla^2 \omega_{ij} &= \omega_{ij,kk} = 0, \\ \omega_{ij,j} &= -\, d_{jj,i}, \\ d_{ij,kk} &= -\, d_{kk,ij}. \end{aligned} \tag{3.5}$$

4. Upper bounds for $K^2_{\mathfrak{G}}$ in terms of an upper bound for $K^M_{\mathfrak{G}}$

That every upper bound for KORN's constant in the main case determines a corresponding upper bound for KORN's constant in the second case was shown for an Ω-domain by FRIEDRICHS [*1*]. We wish to present here a slight variation or refinement of his result which is necessary for us to establish upper bounds for $K^2_{\mathfrak{G}}$ as small as those announced in Section 1.

In this section we take $\mathfrak{G}$ to be an Ω-domain to which the divergence theorem applies. As shown by FRIEDRICHS, any vector field in $\mathscr{C}^1$, satisfying the side conditions $(1.5)_2$ of the second case, may be uniquely decomposed into a sum

$$\boldsymbol{u} = {}'\boldsymbol{u} + {}''\boldsymbol{u}, \tag{4.1}$$

where ${}'\boldsymbol{u} \in \mathscr{D}^\circ$, and ${}''\boldsymbol{u}$ satisfies the side conditions of the main case, namely,

$${}''\boldsymbol{u} \in \mathscr{C}^3, \qquad {}''d_{ij,j} = 0, \qquad \int_{\mathfrak{G}} {}''\omega_{ij}\, dv = 0, \tag{4.2}$$

where ${}''d_{ij}$ and ${}''\omega_{ij}$ are the deformation and rotation measures of ${}''\boldsymbol{u}$. This follows from the known uniqueness, existence, and differentiability properties of solutions of (4.2) with prescribed data for ${}''\boldsymbol{u}$ on the boundary, and the fact that $\int_{\mathfrak{G}} {}'\omega_{ij}\, dv = 0$ for all ${}'\boldsymbol{u}$.

Let K'' be an upper bound for $K^M_{\mathfrak{G}}$, so that we have

$$\int_{\mathfrak{G}} {}''\omega^2\, dv \leqq K'' \int_{\mathfrak{G}} {}''d^2\, dv. \tag{4.3}$$

Now given a field $\boldsymbol{u}\in\mathscr{C}^1$, decomposed as in (4.1), let ${}^\bullet\boldsymbol{u}$ be any field in $\mathscr{C}^2\cap\mathscr{D}^\bullet$ and define

$$*\boldsymbol{u} = {}^\bullet\boldsymbol{u} + {}''\boldsymbol{u}. \tag{4.4}$$

Then

$$\begin{aligned}\int_{\mathfrak{G}} *\omega^2\,dv &= \int_{\mathfrak{G}} [{}^\bullet\omega^2 + {}''\omega^2 + 2\,{}^\bullet\omega_{ij}\,{}''\omega_{ij}]\,dv\\ &= \int_{\mathfrak{G}} [{}^\bullet d^2 - (\operatorname{tr}{}^\bullet d)^2 + K''\,{}''d^2 + 2\,{}^\bullet u_{i,j}\,{}''\omega_{ij}]\,dv.\end{aligned} \tag{4.5}$$

In (4.5) we have used (1.7) to eliminate ${}'\omega^2$ and (4.3) to eliminate ${}''\omega^2$. Using the identity $(3.5)_2$ and the fact that ${}^\bullet\boldsymbol{u}$ vanishes outside some compact subset of $\mathfrak{G}$, we can transform the last term in (4.5), with the aid of the divergence theorem, as follows:

$$\begin{aligned}2\int_{\mathfrak{G}} {}^\bullet u_{i,j}\,{}''\omega_{ij}\,dv &= 2\int_{\mathfrak{G}} [({}^\bullet u_i\,{}''\omega_{ij})_{,j} - {}^\bullet u_i\,{}''\omega_{ij,j}]\,dv\\ &= 2\int_{\mathfrak{G}} {}^\bullet u_i\,d_{jj,i}\,dv = -2\int_{\mathfrak{G}} (\operatorname{tr}{}^\bullet d)\,(\operatorname{tr}{}''d)\,dv\\ &\leqq \int_{\mathfrak{G}} [(\operatorname{tr}{}^\bullet d)^2 + (\operatorname{tr}{}''d)^2]\,dv,\end{aligned} \tag{4.6}$$

where the last line follows from CAUCHY's inequality. Substituting (4.6) into (4.5) and applying (3.3) yields

$$\int_{\mathfrak{G}} *\omega^2\,dv \leqq \int_{\mathfrak{G}} {}^\bullet d^2\,dv + (K''+n)\int_{\mathfrak{G}} {}''d^2\,dv. \tag{4.7}$$

FRIEDRICHS [*1*] has shown that

$$\int_{\mathfrak{G}} *d^2\,dv = \int_{\mathfrak{G}} [{}^\bullet d^2 + {}''d^2]\,dv, \tag{4.8}$$

and thus (4.7) and (4.8) yield

$$\int_{\mathfrak{G}} *\omega^2\,dv \leqq (K''+n)\int_{\mathfrak{G}} *d^2\,dv. \tag{4.9}$$

Now, as remarked by FRIEDRICHS, any ${}^\bullet\boldsymbol{u}\in\mathscr{D}^\bullet$ as well as its first derivatives may be approximated uniformly by a ${}^\bullet\boldsymbol{u}\in\mathscr{D}^\bullet\cap\mathscr{C}^2$. Thus (4.9) holds for all fields (4.4), where ${}^\bullet\boldsymbol{u}\in\mathscr{D}^\bullet$. Moreover, since ${}'\boldsymbol{u}\in\mathscr{D}^\circ$, we may find a ${}^\bullet\boldsymbol{u}\in\mathscr{D}^\bullet$ to make $D(*\boldsymbol{u}-\boldsymbol{u}) = D({}^\bullet\boldsymbol{u}-{}'\boldsymbol{u})$ as small as desired, and since

$$D(\boldsymbol{u}) = \int_{\mathfrak{G}} d^2\,dv + \int_{\mathfrak{G}} \omega^2\,dv$$

for any $\boldsymbol{u}$, it follows from (4.9) that

$$\int_{\mathfrak{G}} \omega^2\,dv \leqq (K''+n)\int_{\mathfrak{G}} d^2\,dv \tag{4.10}$$

for any $\boldsymbol{u}$ satisfying $(1.5)_2$.

Therefore if K'' is an upper bound for KORN's constant in the main case, then K given by

$$K = K'' + n \tag{4.11}$$

is an upper bound for KORN's constant in the second case.

FRIEDRICHS showed that if K'' is an upper bound in the main case, then K given by

$$K = 2\max(1, K'') \tag{4.12}$$

is an upper bound in the second case.

Since we have established that, in three dimensions, $K'' \geqq 3$, the upper bound for the second case as given by (4.11) is never worse than the upper bound (4.12) for any allowed value of K'' if the dimension of the space be three. In two dimensions, since we do not know if $K'' \geqq 2$, we are unable to make a similar comparison.

5. The main case for the sphere and the circle

In this section we consider only vector fields $\boldsymbol{u}$ satisfying the differential equation $d_{ij,j} = 0$ of the main case.

Our proof of a Korn inequality in the main case and our calculation of upper bounds for the Korn constants $K^M_{\mathfrak{S}}$ for the sphere and the circle rest ultimately on the construction of a function ψ defined in the sphere $\mathfrak{S}$ (or circle $\mathfrak{C}$) of radius R with the following properties:

1° $\psi > 0$ in $\mathfrak{S}$ (in $\mathfrak{C}$).

2° ψ and its first derivatives are continuous in $\overline{\mathfrak{S}}$ (in $\overline{\mathfrak{C}}$).

3° There exists a sphere $\mathfrak{S}_\delta$ (a circle $\mathfrak{C}_\delta$) concentric with $\mathfrak{S}$ (with $\mathfrak{C}$) of radius $R_\delta < R$ such that

a) In $\mathfrak{S}_\delta$ (in $\mathfrak{C}_\delta$) ψ is equal to a function which is twice continuously differentiable in $\overline{\mathfrak{S}}_\delta$ (in $\overline{\mathfrak{C}}_\delta$).

b) In $\mathfrak{S} - \overline{\mathfrak{S}}_\delta$ (in $\mathfrak{C} - \overline{\mathfrak{C}}_\delta$) ψ is equal to a function which is twice continuously differentiable in the closure of $\mathfrak{S} - \overline{\mathfrak{S}}_\delta$ (of $\mathfrak{C} - \overline{\mathfrak{C}}_\delta$).

4° $\nabla^2 \psi = h(x)$ in $\mathfrak{S}$ (in $\mathfrak{C}$) where $h = 1$ in $\mathfrak{S} - \mathfrak{S}_\delta$ (in $\mathfrak{C} - \mathfrak{C}_\delta$) and $h = -c_1$ in $\mathfrak{S}_\delta$ (in $\mathfrak{C}_\delta$) where c_1 is a positive constant.

5° ψ and its first derivatives vanish on $\dot{\mathfrak{S}}$, the boundary of $\mathfrak{S}$ (on $\dot{\mathfrak{C}}$, the boundary of $\mathfrak{C}$).

By GREEN's theorem and properties 4° and 5°, it will follow that the mean value of $h(\boldsymbol{x})$ over $\mathfrak{S}$ (over $\mathfrak{C}$) must be zero; hence, the constant c_1 is not at our disposal but is fixed by the radius R_δ.

For the sphere ψ is given by

$$\psi = \tfrac{1}{6}\left[r^2 + 2R^3/r - 3R^2\right] \quad \text{in } \mathfrak{S} - \mathfrak{S}_\delta, \tag{5.1}$$

$$\psi = \tfrac{1}{6}\left[(R_\delta^3 - R^3)\, r^2/R_\delta^3 + 3R^2 (R - R_\delta)/R_\delta\right] \quad \text{in } \mathfrak{S}_\delta. \tag{5.2}$$

For the circle ψ is given by

$$\psi = \tfrac{1}{4}\left[r^2 - R^2 + R^2 \ln (R^2/r^2)\right] \quad \text{in } \mathfrak{C} - \mathfrak{C}_\delta, \tag{5.3}$$

$$\psi = \tfrac{1}{4}\left[(R_\delta^2 - R^2)\, r^2/R_\delta^2 + R^2 \ln (R^2/R_\delta^2)\right] \quad \text{in } \mathfrak{C}_\delta. \tag{5.4}$$

For the remainder of this section, and in Section 6 until (6.18) as well as in Section 7 until (7.9), we add to the requirements $(1.5)_3$ the restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{S}}$ (in $\overline{\mathfrak{C}}$). We remove this restriction after (6.18) and after (7.9).

The objective of the remaining calculations of this section is to obtain inequality (5.19), which we shall use in the following two sections to obtain Korn inequalities for the sphere and circle. We shall understand below that, although we ostensibly are treating the sphere, we may, in the rest of the calculations of this section, replace $\mathfrak{S}$, $\mathfrak{S}_\delta$, *etc.*, by $\mathfrak{C}$, $\mathfrak{C}_\delta$, *etc.*

Although any choice of R_δ between zero and R will yield functions ψ for (5.1) and (5.2) or for (5.3) and (5.4), we shall leave the choice of R_δ arbitrary in this section, for not until Sections 6 and 7 will it be apparent how R_δ may be most advantageously chosen.

Although we are dealing with the main case, not until after equation (5.15) shall we assume that the mean value of the vorticity (ω_{ij}) vanishes, and until then our relations will be independent of this condition, though they will depend on the condition $d_{ij,j}=0$.

From 4° we have

$$\int_{\mathfrak{S}-\mathfrak{S}_\delta} \omega^2\, dv = c_1 \int_{\mathfrak{S}_\delta} \omega^2\, dv + \int_{\mathfrak{S}} h\,\omega^2\, dv. \tag{5.5}$$

By repeated use of the divergence theorem, dropping boundary terms because of 5°, and by using $(3.5)_1$, we get

$$\int_{\mathfrak{S}} h\,\omega^2\, dv = \int_{\mathfrak{S}} \psi_{,kk}\,\omega_{ij}\,\omega_{ij}\, dv = 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv. \tag{5.6}$$

Then adding $\int_{\mathfrak{S}_\delta} \omega^2\, dv$ to both sides of (5.5) and using (5.6), we obtain

$$\int_{\mathfrak{S}} \omega^2\, dv = (1+c_1)\int_{\mathfrak{S}_\delta} \omega^2\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv. \tag{5.7}$$

Using Poincaré's inequality [4] in the form

$$\int_{\mathfrak{S}_\delta} \omega^2\, dv \leqq P_\delta \int_{\mathfrak{S}_\delta} \omega_{ij,k}\,\omega_{ij,k}\, dv + \underset{\delta}{\overline{\omega}}{}^2\, V_\delta, \tag{5.8}$$

where $P_\delta>0$ depends only on the region $\mathfrak{S}_\delta$, V_δ is the volume of $\mathfrak{S}_\delta$, and

$$\underset{\delta}{\overline{\omega}}{}^2 = \underset{\delta}{\overline{\omega}}_{ij}\,\underset{\delta}{\overline{\omega}}_{ij}, \qquad \underset{\delta}{\overline{\omega}}_{ij} = \frac{1}{V_\delta}\int_{\mathfrak{S}_\delta} \omega_{ij}\, dv, \tag{5.9}$$

we get from (5.7)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq (1+c_1)\, P_\delta \int_{\mathfrak{S}_\delta} \omega_{ij,k}\,\omega_{ij,k}\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + (1+c_1)\, V_\delta\, \underset{\delta}{\overline{\omega}}{}^2. \tag{5.10}$$

Now since $\psi>0$ in $\mathfrak{S}$, ψ has a positive minimum in $\overline{\mathfrak{S}}_\delta$, say $c_2 R_\delta^2$. Then

$$1 < \frac{\psi}{c_2 R_\delta^2} \text{ in } \mathfrak{S}_\delta,$$

and hence

$$\begin{aligned} \int_{\mathfrak{S}} \omega^2\, dv &\leqq [(1+c_1)\, P_\delta/c_2 R_\delta^2] \int_{\mathfrak{S}_\delta} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + \\ &\quad + (1+c_1)\, V_\delta\, \underset{\delta}{\overline{\omega}}{}^2 \leqq c_3 \int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + (1+c_1)\, V_\delta\, \underset{\delta}{\overline{\omega}}{}^2, \end{aligned} \tag{5.11}$$

where

$$c_3 = (1+c_1)\, P_\delta/c_2 R_\delta^2 + 2. \tag{5.12}$$

From (5.11) and (3.4) follows

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq 2c_3 \int_{\mathfrak{S}} \psi\,(d_{ij,k}\,d_{ij,k} - d_{ij,k}\,d_{ki,j})\, dv + \underset{\delta}{\overline{\omega}}{}^2\, V_\delta (1+c_1). \tag{5.13}$$

Using the divergence theorem, remembering that $\psi=0$ on $\dot{\mathfrak{S}}$, and $d_{ij,j}=0$, we get from (5.13)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq 2c_3 \int_{\mathfrak{S}} [-\tfrac{1}{2}\psi_{,k}(d^2)_{,k} - \psi\, d_{ij,kk}\, d_{ij} + \psi_{,k}\, d_{ij}\, d_{ki,j}]\, dv + \underset{\delta}{\overline{\omega}}{}^2\, V_\delta(1+c_1). \quad (5.14)$$

Next we use the identity $(3.5)_3$ and the fact that both ψ and $\psi_{,k}$ vanish on $\dot{\mathfrak{S}}$ to obtain from (5.14)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} [\psi_{,kk}\, d^2 + 2\psi_{,ij}(d_{kk}\, d_{ij} - d_{ik}\, d_{kj})]\, dv + \underset{\delta}{\overline{\omega}}{}^2\, V_\sigma(1+c_1). \quad (5.15)$$

Now suppose that $\boldsymbol{u}$ also satisfies the second side condition $(1.5)_3$ of the main case. If we define

$$^*u_i = u_i - x_j\, \underset{\delta}{\overline{\omega}}{}_{ij},$$

we shall have

$$\begin{gathered} ^*d_{ij} = d_{ij}, \qquad ^*\omega_{ij} = \omega_{ij} - \underset{\delta}{\overline{\omega}}{}_{ij}, \\ \int_{\mathfrak{S}_\delta} {}^*\omega_{ij}\, dv = 0. \end{gathered} \quad (5.16)$$

Thus $^*d_{ij,j}=0$, and we may substitute (5.16) into (5.15) to get

$$\int_{\mathfrak{S}} {}^*\omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} [\psi_{,kk}\, d^2 + 2\psi_{,ij}(d_{kk}\, d_{ij} - d_{ik}\, d_{kj})]\, dv. \quad (5.17)$$

But since $(1.5)_3$ holds for ω_{ij},

$$\int_{\mathfrak{S}} \omega^2\, dv = \int_{\mathfrak{S}} {}^*\omega^2\, dv - V\, \underset{\delta}{\overline{\omega}}{}^2 \leqq \int_{\mathfrak{S}} {}^*\omega^2\, dv, \quad (5.18)$$

where V is the volume of $\mathfrak{S}$.

Thus from (5.17) and (5.18) we conclude that

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} [\psi_{,kk}\, d^2 + 2\psi_{,ij}(d_{kk}\, d_{ij} - d_{ik}\, d_{kj})]\, dv, \quad (5.19)$$

which is the relation we sought in this section. An inequality of the form (5.19) holds also for the circle.

6. An upper bound for Korn's constant for the sphere in the main case

We proceed now to use the results of the previous sections to obtain an upper bound for Korn's constant for the sphere in the main case.

To simplify the notation we set

$$\lambda = R/R_\delta, \qquad \lambda > 1. \quad (6.1)$$

For the function ψ defined in (5.1) and (5.2) we have

$$\psi_{,ij} = \tfrac{1}{3}[\delta_{ij}(1 - R^3/r^3) + 3(R^3/r^3)\, n_i\, n_j] \quad \text{in } \mathfrak{S} - \overline{\mathfrak{S}}_\delta \quad (6.2)$$

where

$$n_i = x_i/r, \qquad n_i\, n_i = 1, \quad (6.3)$$

and

$$\psi_{,ij} = -\tfrac{1}{3}(\lambda^3 - 1)\, \delta_{ij} \quad \text{in } \mathfrak{S}_\delta. \quad (6.4)$$

From (6.4) and 4° we have

$$c_1 = -\psi_{,kk} = (\lambda^3 - 1) \quad \text{in } \mathfrak{S}_\delta. \quad (6.5)$$

Now the minimum value of ψ in $\overline{\mathfrak{S}}_\delta$, which is, by definition, $c_2 R_\delta^2$, is assumed on $\dot{\mathfrak{S}}_\delta$ and is given by

$$c_2 R_\delta^2 = \frac{2\lambda^3 - 3\lambda^2 + 1}{6} R_\delta^2. \tag{6.6}$$

Thus c_3 as given in (5.12) has the value

$$c_3 = \frac{6\lambda^3}{2\lambda^3 - 3\lambda^2 + 1} \frac{P_\delta}{R_\delta^2} + 2. \tag{6.7}$$

Insertion of (6.3), (6.4), and (6.5) into (5.19) yields

$$\int_{\mathfrak{S}} \omega^2 dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}_\delta} [-c_1 d^2 - 2(\operatorname{tr} d)^2] dv + \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left[d^2 + 2(\operatorname{tr} d)^2 + 2\frac{R^3}{r^3} A_{ij} n_i n_j\right] dv, \tag{6.8}$$

where

$$A_{ij} = -(\operatorname{tr} d)^2 \delta_{ij} + d^2 \delta_{ij} + 3 d_{kk} d_{ij} - 3 d_{kj} d_{ki}. \tag{6.9}$$

Since the first integral on the right-hand side of (6.8) is negative, we may drop it to obtain, using inequality (3.3) in the second integrand,

$$\int_{\mathfrak{S}} \omega^2 dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left[7 d^2 + \frac{2R^3}{r^3} A_{ij} n_i n_j\right] dv. \tag{6.10}$$

Consider now the form $A_{ij} n_i n_j$ at one point in $\mathfrak{S} - \overline{\mathfrak{S}}_\delta$. If there we diagonalize d_{ij}, getting diagonal components d_1, d_2, d_3, we have, remembering that $n_i n_i = 1$,

$$A_{ij} n_i n_j = d_1 d_2 (1 - 3n_3^2) + d_2 d_3 (1 - 3n_1^2) + d_3 d_1 (1 - 3n_2^2). \tag{6.11}$$

By the method of Lagrange multipliers, we see that the extrema of the form (6.11) are attained when n_i has the form (1, 0, 0), (0, 1, 0), or (0, 0, 1). In the first case, using CAUCHY's inequality, we get

$$\begin{aligned} A_{ij} n_r n_j &= -2 d_1 d_2 + d_2 d_3 + d_3 d_1 \leqq d_1^2 + d_2^2 + \\ &+ \frac{d_2^2 + d_3^2}{2} + \frac{d_3^2 + d_1^2}{2} = \frac{3d_1^2 + 3d_2^2 + 2d_3^2}{2} \leqq \frac{3}{2} d^2. \end{aligned} \tag{6.12}$$

Inequalities similar to (6.12) may be obtained for the other extrema of (6.11), so that for all n_i we have

$$A_{ij} n_i n_j \leqq \tfrac{3}{2} d^2. \tag{6.13}$$

But (6.13) is invariant to orthogonal transformations of the coordinates, and thus it holds for all Cartesian systems.

Putting (6.13) into (6.10) yields

$$\int_{\mathfrak{S}} \omega^2 dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left(7 + 3\frac{R^3}{r^3}\right) d^2 dv. \tag{6.14}$$

Now in $\mathfrak{S} - \mathfrak{S}_\delta$ we have $R^3/r^3 \leqq \lambda^3$, so that from (6.14) we may conclude that

$$\int_{\mathfrak{S}} \omega^2 dv \leqq \frac{c_3}{3} (7 + 3\lambda^3) \int_{\mathfrak{S}-\mathfrak{S}_\delta} d^2 dv \leqq \frac{c_3}{3} (7 + 3\lambda^3) \int_{\mathfrak{S}} d^2 dv. \tag{6.15}$$

A Korn inequality for the sphere in the main case is proven under the restriction that the first and second derivatives of u_i are continuous in $\overline{\mathfrak{S}}$. There remains the problem of determining the best, *i.e.*, the smallest value of the coefficient in (6.15).

The coefficient c_3 in (6.15) is an increasing function of P_δ/R_δ^2 where P_δ is the number introduced into the calculation by our use of POINCARÉ's inequality (5.11). Now it can be shown that, for a sphere,

$$\frac{P_\delta}{R_\delta^2} < \frac{1}{20.19}. \tag{6.16}$$

We say a word about how this number is obtained. It can be shown [4] (Chap. VII, § 7) that P_δ^{-1}, which is the infimum of

$$\frac{\int\limits_{\mathfrak{S}_\delta} f_{,i} f_{,i}\, dv}{\int\limits_{\mathfrak{S}_\delta} f^2\, dv}$$

for all square-integrable functions f with average value zero and with piecewise continuous first derivatives such that $f_{,i} f_{,i}$ is integrable, is the smallest positive number μ^2 for which there exists a square-integrable function g, with piecewise first and second derivatives, such that $g_{,i}\, g_{,i}$ is integrable, satisfying

$$\nabla^2 g + \mu^2 g = 0$$

in $\mathfrak{S}_\delta$, with zero normal derivative on the boundary. By the method of eigenfunction expansions, it can be shown [5] that g is a product of three functions, one each of the three spherical coordinates, and in particular, the radial function, say $\varrho(r)$, satisfies the spherical Bessel equation

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\varrho}{dr} + \left[\mu^2 - \frac{n(n+1)}{r^2}\right]\varrho = 0$$

with n an integer, $d\varrho/dr = 0$ on the boundary and

$$\int\limits_0^{R_\delta} \varrho\, r^2\, dr < \infty, \qquad \int\limits_0^{R_\delta} r^2 (d\varrho/dr)^2\, dr < \infty.$$

From this it follows that μ is the smallest positive number satisfying $\tan(\mu R_\delta) = (\mu R_\delta)$, from which $\mu^2 R_\delta^2 < 20.19$, and hence (6.16).

Using (6.7) and (6.16), we get from (6.15),

$$\begin{aligned} &\int\limits_{\mathfrak{S}} \omega^2\, dv \leq c_4(\lambda) \int\limits_{\mathfrak{S}} d^2\, dv, \\ &c_4(\lambda) = \frac{2}{3}\left[\frac{3\lambda^2}{(20.19)(2\lambda^3 - 3\lambda^2 + 1)} + 2\right](7 + 3\lambda^2). \end{aligned} \tag{6.17}$$

Inequality (6.17) holds for all $\lambda > 1$, since we have left R_δ arbitrary, but smaller than R. Thus, in particular, it will hold for the smallest value of $c_4(\lambda)$, $\lambda > 1$. The minimum of $c_4(\lambda)$, $\lambda > 1$, occurs at about $\lambda = 1.4$ and gives an inequality for the sphere of the form (4.3) for all $''\boldsymbol{u}$ satisfying (4.2) whose second derivatives are continuous in $\overline{\mathfrak{S}}$, with

$$K'' = 17 \tag{6.18}$$

to the best integer.

Now we remove the restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{S}}$, maintaining the side conditions of the main case, $(1.5)_3$.

Let $\mathfrak{S}_\sigma$ be a sphere concentric with $\mathfrak{S}$ and of radius $R_\sigma < R$. Then the first and second derivatives of u_i are continuous in $\overline{\mathfrak{S}}_\sigma$. Thus defining

$$\underset{\sigma}{\overline{\omega}}^2 = \underset{\sigma}{\overline{\omega}}_{ij} \underset{\sigma}{\overline{\omega}}_{ij}, \quad \underset{\sigma}{\overline{\omega}}_{ij} = \frac{1}{V_\sigma} \int_{\mathfrak{S}_\sigma} \omega_{ij} \, dv, \quad V_\sigma = \int_{\mathfrak{S}_\sigma} dv, \tag{6.19}$$

and defining

$$^{\#}u_i = u_i - \underset{\sigma}{\overline{\omega}}_{ij} \, x_j, \tag{6.20}$$

we see that the corresponding strain and rotation measures are given by

$$^{\#}d_{ij} = d_{ij}, \quad ^{\#}\omega_{ij} = \omega_{ij} - \underset{\sigma}{\overline{\omega}}_{ij}, \tag{6.21}$$

and hence, using (6.19) and (6.21),

$$\int_{\mathfrak{S}_\sigma} {}^{\#}\omega_{ij} \, dv = 0. \tag{6.22}$$

Thus from our foregoing analysis we get

$$\int_{\mathfrak{S}_\sigma} {}^{\#}\omega^2 \, dv \leqq K'' \int_{\mathfrak{S}_\sigma} d^2 \, dv \leqq K'' \int_{\mathfrak{S}} d^2 \, dv, \tag{6.23}$$

and hence, using (6.21), (6.22) and (6.23), we obtain

$$\begin{aligned} \int_{\mathfrak{S}} \omega^2 \, dv = \int_{\mathfrak{S}_\sigma} \omega^2 \, dv + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2 \, dv &\leqq \int_{\mathfrak{S}_\sigma} {}^{\#}\omega^2 \, dv + V_\sigma \underset{\sigma}{\overline{\omega}}^2 + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2 \, dv \\ &\leqq K'' \int_{\mathfrak{S}} d^2 \, dv + V \underset{\sigma}{\overline{\omega}}^2 + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2 \, dv. \end{aligned} \tag{6.24}$$

But since $\int_{\mathfrak{S}} \omega_{ij} \, dv = 0$, $\underset{\sigma}{\overline{\omega}}^2$ may be made as small as desired by choosing R_σ close enough to R. Hence (6.23) implies an inequality of the form (4.3) for all $''\boldsymbol{u}_i$ satisfying (4.2) with $K'' = 17$. In other words, 17 is an upper bound for Korn's constant in the main case and, from (4.11), 20 is an upper bound for Korn's constant in the second case. The upper bounds announced in $(1.9)_1$ are established.

7. An upper bound for Korn's constant for the circle in the main case

For the circle, using (5.3) and (5.4), we obtain

$$\psi_{,ij} = \tfrac{1}{2}\left[(2R^2/r^2) \, n_i n_j + (1 - R^2) \, \delta_{ij}\right], \quad \text{in } \mathfrak{C} - \overline{\mathfrak{C}}_\delta \tag{7.1}$$

and

$$\psi_{,ij} = -\tfrac{1}{2}(\lambda^2 - 1) \, \delta_{ij}, \quad \text{in } \mathfrak{C}_\delta. \tag{7.2}$$

From (7.2) and 4° follows

$$c_1 = -\psi_{,kk} = \lambda^2 - 1, \quad \text{in } \mathfrak{C}_\delta. \tag{7.3}$$

As in the case of the sphere, the minimum of ψ in $\overline{\mathfrak{C}}_\delta$ is assumed on $\dot{\mathfrak{C}}_\delta$ and is given by

$$c_2 R_\delta^2 = (\tfrac{1}{4}) \, (1 - \lambda^2 + \lambda^2 \ln \lambda^2) \, R_\delta^2. \tag{7.4}$$

Thus, using (5.12), c_3 becomes

$$c_3 = \frac{4\lambda^2}{1 - \lambda^2 + \lambda^2 \ln \lambda^2} P + 2, \tag{7.5}$$

where $P = P_\delta / R_\delta^2$. It may be shown that P is independent of R_δ, either by simple dimensional considerations or by evaluation of P_δ in a manner similar to that used for the sphere.

Substitution of (7.1), (7.2), and (7.3) into (5.19) yields

$$\int_{\mathfrak{C}} \omega^2\,dv \leqq c_3 \int_{\mathfrak{C}_\delta} \left[-(c_1/2)\,d^2 - (c_1/2)\,(\operatorname{tr} d)^2\right] dv + \\ + c_3 \int_{\mathfrak{C}-\mathfrak{C}_\delta} \left[(\operatorname{tr} d)^2 + (R^2/r^2)\,A_{ij}\,n_i\,n_j\right] dv, \tag{7.6}$$

where

$$A_{ij} = -(\operatorname{tr} d)^2\,\delta_{ij} + d^2\,\delta_{ij} + 2 d_{ij}\,d_{kk} - d_{ik}\,d_{jk}. \tag{7.7}$$

Now to our surprise and delight, $A_{ij} = 0$, which can be shown most easily by referring all quantities to a frame of reference in which d_{ij} is diagonal. Thus, observing that the first integrand on the right-hand side of (7.6) is non-positive and using (3.3), we get

$$\int_{\mathfrak{C}} \omega^2\,dv \leqq \left[\frac{8\lambda^2 P}{1-\lambda^2+\lambda^2 \ln \lambda^2} + 4\right] \int_{\mathfrak{C}} d^2\,dv. \tag{7.8}$$

Since (7.8) holds for all $\lambda > 1$, it holds as $\lambda \to \infty$, in which case we get

$$\int_{\mathfrak{C}} \omega^2\,dv \leqq 4 \int_{\mathfrak{C}} d^2\,dv. \tag{7.9}$$

The restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{C}}$ may be removed in the same manner as for the sphere.

Thus we have proven that 4 is an upper bound for KORN's constant for the circle in the main case and, by (4.11), that 6 is an upper bound in the second case. All of the bounds announced in (1.9) are now proven.

We close with some remarks on the extension of the above analysis for the circle to more general plane regions, namely, those onto which $\mathfrak{C}$ is mapped conformally by a function $\zeta(z) = \xi + i\eta$, $z = x + iy$, which is regular in $\overline{\mathfrak{C}}$. If we denote such a region by $\zeta(\mathfrak{C})$ and express ψ, defined in the x, y plane by (5.3) and (5.4), as a function of ξ and η, we may note the following:

a) The boundary of $\zeta(\mathfrak{C})$ is $\zeta(\dot{\mathfrak{C}})$.

b) Since $\psi_{,\xi}$ and $\psi_{,\eta}$ are linear combinations of $\psi_{,x}$ and $\psi_{,y}$, it follows that ψ and its derivatives are both zero on the boundary of $\zeta(\mathfrak{C})$.

c) The Laplacian of ψ transforms according to

$$\underset{z}{\nabla^2} \psi = |\zeta'(z)|^2 \underset{\xi}{\nabla^2} \psi.$$

d) Because of the regularity of $\zeta(z)$ in $\overline{\mathfrak{C}}$, $|\zeta'(z)|^2$ has a positive maximum and minimum in $\overline{\mathfrak{C}}$. Thus there are positive numbers, say b and B, b_δ and B_δ such that

$$b < \nabla^2 \psi < B \quad \text{in} \quad \zeta(\mathfrak{C} - \mathfrak{C}_\delta)$$
$$-B_\delta < \nabla^2 \psi < -b_\delta \quad \text{in} \quad \zeta(\mathfrak{C}_\delta).$$

Thus the function ψ defined in the ξ, η plane satisfies conditions 1°, 2°, 3°, and 5°, but 4° must be replaced by $d°$. Using $\psi(\xi, \eta)$, then, we may carry through an analysis for $\zeta(\mathfrak{C})$ similar to that in Section 5, except that equations and inequalities which follow from 4° must be replaced by equations and inequalities implied by $d°$.

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On the Global Asymptotic Behavior of Nonlinear Systems of Differential Equations

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1. Introduction

We consider the real system

$$\dot{x} = p(t, x, y), \quad \dot{y} = q(t, x, y) \qquad (\dot{} = d/dt), \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$. Our primary purpose is to find conditions under which all solutions of (1.1) tend to zero as $t \to \infty$. When this situation prevails we shall, for brevity, speak of (1.1) as being "globally asymptotically stable" even though this terminology usually denotes a stronger assertion about the solutions of (1.1). (Note, for example, that $x = 0$, $y = 0$ is not assumed to be a solution of (1.1).) The conditions that imply global asymptotic stability will arise quite naturally from an investigation of the more general problem of the asymptotic behavior (as $t \to \infty$) of all solutions of (1.1).

The splitting of (1.1) into two parts anticipates the key hypothesis. It is assumed that there exists an "energy" function $E(t, x, y)$ which is positive definite on R_{n+m} (see § 2 for terminology) and whose total derivative with respect to (1.1), denoted by $E'(t, x, y)$ where

$$E'(t, x, y) = E_t(t, x, y) + \sum_{i=1}^{n} E_{x_i}(t, x, y)\, p_i(t, x, y) + \sum_{j=1}^{m} E_{y_j}(t, x, y)\, q_j(t, x, y), \tag{1.2}$$

satisfies

$$E'(t, x, y) \leqq -W(y) + e_1(t) + e_2(t)\, E(t, x, y), \tag{1.3}$$

where $W(y)$ is positive definite on R_m and the $e_k(t) \in L_1(0, \infty)$. The $e_k(t)$ are introduced in (1.3) in order to permit the existence of integrable forcing or perturbation terms in (1.1). It is seen from (1.3), in the case $e_k(t) \equiv 0$ $(k = 1, 2)$, that $E'(t, x, y)$ is (in general) negative definite only on a subspace of R_{n+m}.

The existence of an energy function which is positive definite on the entire phase space and whose total derivative is negative definite (or almost negative definite, in the sense of (1.3)) only on a subspace is precisely the situation that occurs in several applications. An important example is the Liénard equation

$$\ddot{x} + h(t, x, \dot{x})\,\dot{x} + f(x) = e(t), \tag{1.4}$$

where h is a positive damping coefficient; f is a spring, that is $x f(x) > 0$ $(x \neq 0)$; and e an integrable forcing term. If one writes (1.4) as the equivalent system

$$\dot{x} = y, \quad \dot{y} = -h(t, x, y)\, y - f(x) + e(t) \tag{1.5}$$

and introduces the energy function

$$E(x, y) = g(x) + \tfrac{1}{2} y^2 \quad \left(g(x) = \int_0^x f(\xi)\, d\xi,\right),$$

then

$$E'(t, x, y) = -h(t, x, y)\, y^2 + y\, e(t).$$

If $h \geqq k > 0$, where k is a constant, then it is easily shown that

$$E'(t, x, y) \leqq -W(y) + |e(t)| + 2|e(t)|\, E(x, y),$$

where $W(y) = k y^2$.

In terms of the autonomous case of (1.1),

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1.6}$$

the main lines of the argument are as follows. Under appropriate hypotheses, including local existence but not uniqueness of solutions of (1.6), it is shown for any x_0, y_0 that any solution $x(t)$, $y(t)$ of (1.6) satisfying $x(0) = x_0$, $y(0) = y_0$: exists on $0 \leqq t < \infty$; is bounded there, the bound depending only on x_0, y_0; and $y(t) \to 0$ as $t \to \infty$. With $x(t)$ thought of as a curve in R_n, it is then shown that the cluster points of $x(t)$ as $t \to \infty$ are contained in the set Ω_1 defined by

$$\Omega_1 = \{x \mid q(x, 0) = 0\}. \tag{1.7}$$

(If, in a particular case, Ω_1 contains only the origin, then it necessarily follows that $x(t) \to 0$ as $t \to \infty$. This is easily seen to be the situation for the autonomous case of (1.5), where $q(x, 0) = -f(x)$ and $f(x) = 0$ implies $x = 0$.) Finally, with the aid of the system

$$\dot{\varphi} = p(\varphi, 0), \tag{1.8}$$

it is shown that the cluster points of $x(t)$ are contained in a set Ω, which is independent of x_0, y_0 and is (in general) a proper subset of Ω_1. In particular, any point of Ω_1 which is ever mapped out of Ω_1 by a solution of (1.8) cannot be a cluster point of $x(t)$. Alternatively, the cluster points of $x(t)$ are contained in the set of points, Ω, through which there passes a solution of

$$\dot{\varphi} = p(\varphi, 0), \quad 0 = q(\varphi, 0). \tag{1.9}$$

Thus, taken together with the other hypotheses, a sufficient condition for the global asymptotic stability of (1.6) is that $\varphi(t) \equiv 0$ be the only solution of (1.9).

Besides (1.4), other interesting examples for which the present methods apply occur in the stability theory of heterogeneous nuclear reactors, see [*4*]. Once the appropriate energy functions are shown to exist, Corollaries 1 and 2 below contain the results of [*4*] as very special cases. When restricted to the systems of [*4*], Corollaries 1 and 2 also improve the earlier results in that less smoothness hypothesis is required, *e.g.*, the Lipschitz condition imposed on the $f(x)$ of (1.4) in [*4*] may be replaced by continuity. In fact, it may easily be shown that Corollary 1 also permits the $f(x)$ term in (1.4) to depend suitably on t.

It should be observed that the crucial hypotheses of Corollaries 1 and 2, $0=\Omega_1$ and $0=\Omega$ respectively, can often be verified easily by inspection. The systems of [*4*], as well as many other easily constructed examples, are of this type.

Lemma 3.1 is a boundedness result for nonlinear systems of differential equations. When restricted to (1.4), it is an improvement of a theorem due to ANTOSIEWICZ [*1*]. If $n=0$, *i.e.*, if (1.1) is replaced by $\dot{y}=q(t,y)$, then Lemmas 3.1 and 3.2 supplement Theorem 1 of ANTOSIEWICZ [*2*].

The technique of analyzing the stability properties of differential equations by means of energy functions is often called LYAPUNOV's second method. A recent survey of this method is [*3*]; see § 4 of [*3*], where there is some discussion of global asymptotic behavior.

2. Summary

The following notation is used: $I_{t_0}=\{t\,|\,t_0\leqq t<\infty\}$, where $t_0\geqq 0$. $I=I_0$. R_k, k-dimensional Euclidean space. $|x|=\sum|x_i|$ for $x\in R_k$. d, the metric induced on R_n by the norm $|\ |$. $K, K_i, K(\ )$, finite positive real numbers (which may vary from line to line). $S_k^K=\{x\,|\,x\in R_k,\ |x|\leqq K\}$. $C(\ )$, $C'(\ )$, $\mathrm{Lip}(\ )$, the sets of (vector) functions that are, respectively, continuous, continuously differentiable with respect to each variable, satisfy a Lipschitz condition (on the appropriate space). $L_1(I)$, the set of Lebesgue integrable functions on I. A scalar function $F(x)$ is said to be positive definite on R_k if $F(x)>0$ for all $x\neq 0$ and $F(0)\geqq 0$. $F(t,x)$ is said to be positive definite on R_k if there exists an $F_1(x)$ which is positive definite on R_k and $F(t,x)\geqq F_1(x)$.

It is assumed that p and q may be written as

$$p=p_\alpha+p_\beta,\qquad q=q_\alpha+q_\beta,$$

where p_β and q_β may be thought of as integrable forcing or perturbation terms, and where the following smoothness and growth conditions are satisfied:

(2.1) $$p,q\in C(I\times R_{n+m}).$$

For every K:

(2.2) *p is bounded on $I\times S_{n+m}^K$;*

(2.3) *q is bounded on $I\times S_{n+m}^K$;*

(2.4α) *there exists $\hat{p}_\alpha(x)\in\mathrm{Lip}(S_n^K)$ such that*

$$\lim_{t\to\infty,\,y\to 0} p_\alpha(t,x,y)-\hat{p}_\alpha(x),$$

where the limit is taken in t and y together, and is assumed to be uniform with respect to x on S_n^K;

(2.4β) *there exists a scalar function $\hat{p}_\beta(t;K)\in L_1(I)$ such that*

$$|p_\beta(t,x,y)|\leqq\hat{p}_\beta(t;K)\qquad((t,x,y)\in I\times S_{n+m}^K);$$

(2.5α) *there exists $\hat{q}_\alpha(x)\in C(R_n)$ such that*

$$\lim_{t\to\infty,\,y\to 0} q_\alpha(t,x,y)=\hat{q}_\alpha(x),$$

where the limit is taken in t and y together, and is assumed to be uniform with respect to x on S_n^K;

(2.5 β) *there exists a scalar function* $\hat{q}_\beta(t; K) \in L_1(I)$ *such that*

$$|q_\beta(t, x, y)| \leqq \hat{q}_\beta(t; K) \qquad ((t, x, y)\ I \times S^K_{n+m}).$$

It should be noted that in the autonomous case $p_\beta = 0$, $q_\beta = 0$ so that (2.4 β, 2.5 β) are trivially satisfied; (2.1) implies (2.2, 2.3, 2.5 α); and $p \in C'(R_{n+m})$ is sufficient for (2.4 α).

Concerning the energy function $E(t, x, y)$ and the associated $E'(t, x, y)$ defined by (1.2) it is assumed that:

(2.6) *There exists* $E(t, x, y) \in C'(I \times R_{n+m})$ *and a positive definite* $E_1(x, y) \in C(R_{n+m})$ *such that* $E_1(x, y) \leqq E(t, x, y)$ *and* $E_1(x, y) \to \infty$ *as* $|x| + |y| \to \infty$. *Furthermore,* $E(t, x, y)$ *is bounded on* $I \times S^K_{n+m}$ *for every* K *and* $E(t, 0, 0) = 0$.

(2.7) *There exist* $e_k(t) \in L_1(I)$ $(k = 1, 2)$ *such that*

$$E'(t, x, y) \leqq e_1(t) + e_2(t)\, E(t, x, y).$$

(2.8) *There exist* $e_k(t) \in L_1(I)$ $(k = 1, 2)$ *and a positive definite* $W(y) \in C(R_m)$ *such that* $W(0) = 0$ *and*

$$E'(t, x, y) \leqq -W(y) + e_1(t) + e_2(t)\, E(t, x, y).$$

Although (2.7) is implied by (2.8), it is listed separately, since it is sufficient for Lemma 3.1. We shall assume, without loss of generality, that the $e_k(t)$ of (2.7, 2.8) are non-negative.

Let $(t_0, x_0, y_0) \in I \times R_{n+m}$. By $\mathscr{S} = \mathscr{S}(t, t_0, x_0, y_0)$ we mean any solution $x(t)$, $y(t)$ of (1.1) that exists on some subinterval of I having t_0 as left-hand end point and that satisfies $x(t_0) = x_0$, $y(t_0) = y_0$. It is seen from (2.1) that for any given (t_0, x_0, y_0) an $\mathscr{S}$ exists but is not necessarily unique. $\omega \in R_n$ is called a cluster point of $\mathscr{S}$, for an $\mathscr{S}$ existing on I_{t_0}, if there exists a sequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that $x(t_k) \to \omega$ as $k \to \infty$. More properly, we might call such an $\omega \in R_n$ a cluster point of $x(t)$ rather than of $\mathscr{S}$. The definition anticipates that, under the preceding conditions, for any $\mathscr{S}$, $y(t) \to 0$ as $t \to \infty$, as will be shown. Thus, $(x, y) = (\omega, 0)$ is really a cluster point of $\mathscr{S}$. The set of cluster points of $\mathscr{S}$ is denoted by $\mathscr{S}^+$.

Central to our considerations is the subset Ω_1 of R_n defined by

$$\Omega_1 = \{x \mid \hat{q}_\alpha(x) = 0\}. \tag{2.9}$$

From the continuity of $\hat{q}_a$ it follows that Ω_1 is closed.

Theorem 1. *Let* (2.1, 2.2, 2.3, 2.5, 2.6, 2.8) *be satisfied, and let* $(t_0, x_0, y_0) \in I \times R_{n+m}$. *Then any* $\mathscr{S} = \mathscr{S}(t, t_0, x_0, y_0)$ *exists on* I_{t_0} *and* $(x(t), y(t)) \in S^K_{n+m}$ *for some* $K = K(t_0, x_0, y_0)$. *Furthermore,* $y(t) \to 0$ *as* $t \to \infty$ *and* $\mathscr{S}^+ \subset \Omega_1$.

In Lemma 3.4 it is shown that $0 \in \Omega_1$. The following result on global asymptotic stability is an immediate consequence of Theorem 1.

Corollary 1. *If in addition to the hypothesis of Theorem 1 it is further assumed that* $0 = \Omega_1$, *then it also follows that* $x(t) \to 0$ *as* $t \to \infty$.

The conclusion $\mathscr{S}^+ \subset \Omega_1$ of Theorem 1 can be considerably improved if condition (2.4) is also assumed. For this purpose, we consider the system

$$\dot{\varphi} = \hat{p}_\alpha(\varphi). \tag{2.10}$$

Let $\varphi(t, \xi)$ denote the solution of (2.10) satisfying $\varphi(0, \xi) = \xi$. It is well known, note the Lipschitz condition of (2.4α), that the mapping M_t of R_n into R_n defined by

$$M_t \xi = \varphi(t, \xi) \qquad (\xi \in R_n) \tag{2.11}$$

is locally a homeomorphism. The set $\Omega_2 \subset \Omega_1$ is defined by

$$\Omega_2 = \{x \mid x \in \Omega_1 \quad \text{and, for some} \quad t = t(x) > 0,\ M_t x \notin \Omega_1\}. \tag{2.12}$$

Define

$$\Omega = \Omega_1 - \Omega_2. \tag{2.13}$$

It is clear from the preceding definitions that Ω may be equivalently described as the set of points in R_n through which there passes a solution of

$$\dot{\varphi} = \hat{p}_\alpha(\varphi), \qquad 0 = \hat{q}_\alpha(\varphi). \tag{2.14}$$

Theorem 2. *Let the hypothesis of Theorem 1 as well as (2.4) be satisfied. Then it also follows that $\mathscr{S}^+ \subset \Omega$.*

In Lemma 4.3 it is shown that $0 \in \Omega$. The following result on global asymptotic stability is an immediate consequence of Theorem 2.

Corollary 2. *If in addition to the hypothesis of Theorem 2 it is further assumed that $0 = \Omega$, then it also follows that $x(t) \to 0$ as $t \to \infty$.*

3. Proof of Theorem 1

Lemmas 3.1, 3.2, and 3.3 establish Theorem 1. Lemma 3.4 shows that $0 \in \Omega_1$ as asserted in § 2.

Lemma 3.1. *Let (2.1, 2.6, 2.7) be satisfied, and let $(t_0, x_0, y_0) \in I \times R_{n+m}$. Then any $\mathscr{S} = \mathscr{S}(t, t_0, x_0, y_0)$ exists on I_{t_0}, and*

$$|x(t)| \leqq K, \qquad |y(t)| \leqq K \qquad (t \in I_{t_0}) \tag{3.1}$$

for some $K = K(t_0, x_0, y_0)$.

Proof. By (2.1) there is a $t_1 = t_1(\mathscr{S}) > t_0$ such that $\mathscr{S}$ exists on $t_0 \leqq t \leqq t_1$. Define

$$\begin{gathered} E(t) = E\big(t, x(t), y(t)\big), \quad E_1(t) = E_1\big(x(t), y(t)\big), \\ E'(t) = E'\big(t, x(t), y(t)\big) \qquad (t_0 \leqq t \leqq t_1). \end{gathered} \tag{3.2}$$

Then (1.2, 2.7) imply

$$\dot{E}(t) = E'(t) \leqq e_1(t) + e_2(t)\, E(t) \qquad (t_0 \leqq t \leqq t_1),$$

and thus

$$E(t) \leqq E(t_0) + \int_{t_0}^{\infty} e_1(\tau)\, d\tau + \int_{t_0}^{t} e_2(\tau)\, E(\tau)\, d\tau \qquad (t_0 \leqq t \leqq t_1).$$

A well known inequality now yields

$$E_1(t) \leqq E(t) \leqq \left\{E(t_0) + \int_{t_0}^{\infty} e_1(\tau)\, d\tau\right\} \exp\left[\int_{t_0}^{\infty} e_2(\tau)\, d\tau\right] \qquad (t_0 \leqq t \leqq t_1). \tag{3.3}$$

Upon noting that the right-hand side of (3.3) is finite and independent of t_1, the conclusion follows readily from (2.6, 2.7, 3.3) and a routine continuation argument.

Lemma 3.2. *Let* (2.1, 2.3, 2.6, 2.8) *be satisfied, and let* $\mathscr{S}$ *be as in Lemma* 3.1. *Then*

$$\lim_{t\to\infty} y(t) = 0. \tag{3.4}$$

Proof. Define $E(t)$, $E'(t)$ on I_{t_0} by (3.2). Then by (1.2, 2.8)

$$\dot{E}(t) = E'(t) \leqq -W\big(y(t)\big) + e_1(t) + e_2(t)\,E(t),$$

from which

$$0 \leqq E(t) \leqq E(t_0) - \int_{t_0}^{t} W\big(y(\tau)\big)\,d\tau + \int_{t_0}^{t} e_1(\tau)\,d\tau + \int_{t_0}^{t} e_2(\tau)\,E(\tau)\,d\tau. \tag{3.5}$$

It follows readily from (2.6, 2.8, 3.1, 3.5) that $W\big(y(t)\big) \in L_1(I_{t_0})$.

Suppose (3.4) is not true. From (1.1, 2.3, 3.1) one has $|\dot{y}(t)| \leqq K\,(t \in I_{t_0})$ for some K. Hence, there exists an $\varepsilon_1 > 0$ and a sequence of intervals $\{(t_n, t_n + \lambda)\}$, where $t_n \to \infty$ as $n \to \infty$ and $\lambda > 0$, such that

$$|y(t)| \geqq \varepsilon_1 \qquad (t_n \leqq t \leqq t_n + \lambda;\ n = 1, 2, \ldots). \tag{3.6}$$

But from (2.8, 3.1, 3.6) one has for some $\varepsilon_2 > 0$

$$W\big(y(t)\big) \geqq \varepsilon_2 \qquad (t_n \leqq t \leqq t_n + \lambda;\ n = 1, 2, \ldots),$$

which contradicts $W\big(y(t)\big) \in L_1(I_{t_0})$ and completes the proof.

Lemma 3.3. *Let* (2.1, 2.2, 2.3, 2.5, 2.6, 2.8) *be satisfied, and let* $\mathscr{S}$ *be as in Lemma* 3.1. *Then*

$$\mathscr{S}^+ < \Omega_1. \tag{3.7}$$

Proof. Let $\lambda > 0$ and $t \in I_{t_0}$. Then from (1.1) one has

$$y(t+\lambda) - y(t) = \int_t^{t+\lambda} \{q_\alpha\big(\tau, x(\tau), y(\tau)\big) - \hat{q}_\alpha\big(x(\tau)\big)\}\,d\tau + \int_t^{t+\lambda} \hat{q}_\alpha\big(x(\tau)\big)\,d\tau + \\ + \int_t^{t+\lambda} q_\beta\big(\tau, x(\tau), y(\tau)\big)\,d\tau,$$

which together with (2.5, 3.1, 3.4) easily implies

$$\lim_{t\to\infty} \int_t^{t+\lambda} \hat{q}_\alpha\big(x(\tau)\big)\,d\tau = 0. \tag{3.8}$$

We wish to show that

$$\lim_{t\to\infty} \hat{q}_\alpha\big(x(t)\big) = 0. \tag{3.9}$$

Suppose (3.9) is false. Then it is false for some component, say $\hat{q}_{\alpha,i}\big(x(t)\big)$. Then there exists an $\varepsilon_1 > 0$ and a sequence $\{t_k\}$, where $t_k \to \infty$ as $k \to \infty$, such that $|\hat{q}_{\alpha,i}\big(x(t_k)\big)| \geqq \varepsilon_1$ $(k = 1, 2, \ldots)$. We assume that $\hat{q}_{\alpha,i}\big(x(t_k)\big) \geqq \varepsilon_1$, a similar argument holds for $\hat{q}_{\alpha,i}\big(x(t_k)\big) \leqq -\varepsilon_1$. Uniform continuity, which one has from (2.5 α, 3.1), implies the existence of a $\delta > 0$ and independent of k such that

$$\hat{q}_{\alpha,i}(\xi) \geqq \tfrac{1}{2}\varepsilon_1 \qquad \big(|\xi - x(t_k)| \leqq \delta;\ k = 1, 2, \ldots\big). \tag{3.10}$$

However, by (1.1, 2.2, 3.1, 3.10) it follows that there exists a $\lambda_1 > 0$ and independent of k such that

$$\hat{q}_{\alpha,i}(x(t)) \geqq \tfrac{1}{2}\varepsilon_1 \qquad (t_k \leqq t \leqq t_k + \lambda_1;\ k = 1, 2, \ldots). \tag{3.11}$$

Clearly (3.11) contradicts (3.8) for $\lambda = \lambda_1$. Hence, (3.9) is true, and this easily implies (3.7).

Lemma 3.4. *Let* (2.1, 2.2, 2.3, 2.5, 2.6, 2.8) *be satisfied. Then* $\hat{q}_\alpha(0) = 0$, *or, equivalently*, $0 \in \Omega_1$.

Proof. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Then by the continuity of $\hat{q}_\alpha$ it suffices to show that there exists a $\xi \in R_n$ such that $|\xi| < \varepsilon_1$ and $|\hat{q}_\alpha(\xi)| < \varepsilon_2$. By (2.6, 3.3) t_0 may be chosen sufficiently large so that for any $\mathscr{S} = \mathscr{S}(t, t_0, 0, 0)$ it follows that $|x(t)| < \varepsilon_1$ for $t \in I_{t_0}$. From (3.9) one has, however, the existence of a $t_1 \geqq t_0$ such that $|\hat{q}_\alpha(x(t_1))| < \varepsilon_2$. The result follows immediately on choosing $\xi = x(t_1)$.

4. Proof of Theorem 2

Lemmas 4.1 and 4.2 establish Theorem 2. Lemma 4.3 shows that $0 \in \Omega$ as asserted in § 2.

Lemma 4.1. *Let* (2.1, 2.3, 2.4, 2.6, 2.8) *be satisfied, and let* $\mathscr{S}$ *be as in Lemma* 3.1. *Let* $\varepsilon > 0$ *and* $\lambda > 0$. *Then there exists an* $\eta(\varepsilon, \lambda, t_0, x_0, y_0) > 0$ *and a* $T_1(\varepsilon, \lambda, \mathscr{S}) \in I_{t_0}$ *such that* $t_1 \in I_{T_1}$, $\xi \in R_n$ *and satisfying*

$$|x(t_1) - \xi| < \eta \tag{4.1}$$

implies

$$|x(t) - M_{t-t_1}\xi| < \varepsilon \qquad (t_1 \leqq t \leqq t_1 + \lambda). \tag{4.2}$$

Proof. By (3.1) there is a $K_1 > 0$ such that $|x(t)|, |y(t)| \leqq K_1$ for $t \in I_{t_0}$. Without loss of generality we assume that $\varepsilon < K_1$. Let K_2 denote the Lipschitz constant in $S_n^{2K_1}$ of (2.4α) for $\hat{p}_\alpha$. Choose $\eta > 0$ so that

$$\eta \leqq \tfrac{1}{6}\varepsilon \exp[-K_2\lambda]. \tag{4.3}$$

By (2.4, 3.1, 3.4) we can choose $T_1 \in I_{t_0}$ such that

$$\begin{aligned} &|p_\alpha(t, x(t), y(t)) - \hat{p}_\alpha(x(t))| \leqq \frac{\varepsilon}{6\lambda} \exp[-K_2\lambda] \qquad (t \in I_{T_1}), \\ &\int_{T_1}^{\infty} \hat{p}_\beta(t; K_1)\, dt \leqq \tfrac{1}{6}\varepsilon \exp[-K_2\lambda]. \end{aligned} \tag{4.4}$$

Let $t_1 \in I_{T_1}$, and let ξ satisfy (4.1). Using (2.11), define $\varphi(t)$ by $\varphi(t) = M_{t-t_1}\xi = \varphi(t - t_1, \xi)$. As $\hat{p}_\alpha$ is continuous there exists a t_2, where $t_1 < t_2 \leqq t_1 + \lambda$, such that $|\varphi(t)| \leqq 2K_1$ for $t_1 \leqq t \leqq t_2$. Since (2.10) is autonomous, one has from (1.1, 2.10)

$$\begin{aligned} x(t) - \varphi(t) = x(t_1) - \xi + \int_{t_1}^{t} \{p_\alpha(\tau, x(\tau), y(\tau)) - \hat{p}_\alpha(x(\tau))\}\, d\tau\, + \\ + \int_{t_1}^{t} \{\hat{p}_\alpha(x(\tau)) - \hat{p}_\alpha(\varphi(\tau))\}\, d\tau + \int_{t_1}^{t} p_\beta(\tau, x(\tau), y(\tau))\, d\tau \qquad (t_1 \leqq t \leqq t_2), \end{aligned}$$

which together with (2.4, 4.3, 4.4) yields

$$|x(t)-\varphi(t)| \leqq \tfrac{1}{2}\varepsilon \exp[-K_2\lambda] + K_2\int_{t_1}^{t}|x(\tau)-\varphi(\tau)|\,d\tau \qquad (t_1 \leqq t \leqq t_2).$$

A well known inequality now implies

$$|x(t)-\varphi(t)| \leqq \tfrac{1}{2}\varepsilon \exp[K_2(t-t_1-\lambda)] \qquad (t_1 \leqq t \leqq t_2), \tag{4.5}$$

which in turn implies

$$|\varphi(t)| \leqq K_1 + \tfrac{1}{2}\varepsilon \exp[K_2(t-t_1-\lambda)] \qquad (t_1 \leqq t \leqq t_2). \tag{4.6}$$

The conclusion follows readily from (4.5, 4.6).

Lemma 4.2. *Let* (2.1—2.6, 2.8) *be satisfied, and let* $\mathscr{S}$ *be as in Lemma* 3.1. *Then* $\mathscr{S}^+ \subset \Omega$.

Proof. Suppose there exists an $\omega \in \mathscr{S}^+$ but $\omega \notin \Omega$. Then by (2.13, 3.7) one has $\omega \in \mathscr{S}^+ \cap \Omega_2$. Since $\omega \in \mathscr{S}^+$, there exists a sequence $\{t_k\}$, with $t_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} x(t_k) = \omega. \tag{4.7}$$

Since $\omega \in \Omega_2$, there exists (see (2.12)) a $\lambda > 0$ such that $M_\lambda \omega \notin \Omega_1$. Hence, it follows, as Ω_1 is closed, that $d(M_\lambda\omega, \Omega_1) = \varrho > 0$. The homeomorphic nature of M_t then implies that there exists a ν, where $0 < \nu < \frac{1}{2}\varrho$, such that

$$d(M_\lambda\zeta, \Omega_1) \geqq \tfrac{1}{2}\varrho \qquad (|\zeta-\omega| < \nu). \tag{4.8}$$

Choose N sufficiently large so that by (4.7), (3.7), and Lemma 4.1 (with $\xi = x(t_N)$) one has respectively

$$|x(t_N)-\omega| < \tfrac{1}{4}\nu, \tag{4.9}$$

$$d(x(t), \Omega_1) < \tfrac{1}{4}\nu, \qquad (t \in I_{t_N}), \tag{4.10}$$

$$|x(t) - M_{t-t_N}x(t_N)| < \tfrac{1}{4}\nu \qquad (t_N \leqq t \leqq t_N+\lambda). \tag{4.11}$$

From the triangle inequality

$$d(M_\lambda x(t_N), \Omega_1) \leqq |M_\lambda x(t_N) - x(t_N+\lambda)| + d(x(t_N+\lambda), \Omega_1)$$

(which is easily shown to be valid even though one "point" is Ω_1), (4.8, 4.9), and (4.10, 4.11) at $t = t_N + \lambda$ one has $\frac{1}{2}\varrho \leqq \frac{1}{4}\nu + \frac{1}{4}\nu$, which contradicts $\nu < \frac{1}{2}\varrho$ and completes the proof.

Lemma 4.3. *Let* (2.1—2.6, 2.8) *be satisfied. Then* $0 \in \Omega$.

Proof. By (2.12, 2.13) and Lemma 3.4 it is sufficient to show that $\varphi(t, 0) = 0$ for $t \in I$. This is, as (2.10) is autonomous and $\hat{p}_\alpha$ satisfies a Lipschitz condition, equivalent to $\hat{p}_\alpha(0) = 0$. Hence, it suffices to show that $\varphi(t, 0) = 0$ on $0 \leqq t \leqq \lambda$ for some $\lambda > 0$. Let $\lambda > 0$ and $\nu > 0$ be sufficiently small so that by (2.4α) $\varphi(t, \xi)$ exists on $0 \leqq t \leqq \lambda$, $|\xi| < \nu$.

Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, with $\varepsilon_1 < \nu$. By continuity it suffices to show that there exist a $\xi \in R_n$ such that $|\xi| < \varepsilon_1$ and $|\varphi(t, \xi)| < \varepsilon_2$ on $0 \leqq t \leqq \lambda$. By (2.6, 3.3) t_0 may be chosen sufficiently large so that for any $\mathscr{S} = \mathscr{S}(t, t_0, 0, 0)$ it follows that $|x(t)| < \min(\varepsilon_1, \frac{1}{3}\varepsilon_2)$ for $t \in I_{t_0}$. From (4.2) one has the existence of a $t_1 \geqq t_0$ such

that
$$|x(t) - \varphi(t - t_1, x(t_1))| < \tfrac{1}{3}\varepsilon_2 \qquad (t_1 \leqq t \leqq t_1 + \lambda),$$
which implies
$$|\varphi(t - t_1, x(t_1))| < \tfrac{2}{3}\varepsilon_2 \qquad (t_1 \leqq t \leqq t_1 + \lambda),$$
and, hence
$$|\varphi(t, x(t_1))| < \tfrac{2}{3}\varepsilon_2 \qquad (0 \leqq t \leqq \lambda).$$
The result follows immediately on choosing $\xi = x(t_1)$.

5. Supplementary Remarks

While (5.1, 5.2) below are essentially implicit in the preceding sections, the following modification of the proofs yields a little more information. This modification has been pointed out to us by Y. Sibuya. Lemmas 3.1 and 3.2 are established as above. Then, let $\omega \in \mathscr{S}^+$ for some $\mathscr{S} = \mathscr{S}(t, t_0, x_0, y_0)$, and let $x(t_k) \to \omega$ as $k \to \infty$, where $t_k \to \infty$ as $k \to \infty$. By an argument similar to that of Lemma 4.1 it may be shown that $\varphi(t, \omega)$ exists on I and that

$$\lim_{k \to \infty} x(t + t_k) = \varphi(t, \omega) \tag{5.1}$$

uniformly with respect to t on any finite t interval $0 \leqq t \leqq T$. Then by (5.1) and the same calculation used to establish (3.8) one can easily show that

$$\int_0^t \hat{q}_\alpha(\varphi(\tau, \omega))\, d\tau = 0 \qquad (t \in I)$$

and, hence, that

$$\hat{q}_\alpha(\varphi(t, \omega)) = 0 \qquad (t \in I). \tag{5.2}$$

Clearly (5.2) implies that $\mathscr{S}^+ \subset \Omega$.

The method of the above paragraph has the advantage of showing that (2.2) is not a necessary hypothesis for Theorem 2. However, it appears that (2.2) is necessary for Theorem 1 unless, of course, (2.4) is assumed, in which case Theorem 1 is a trivial consequence of Theorem 2. It may also be noted that (2.3) cannot be entirely dispensed with, as has been shown in Remark 2.2 of [*4*].

As alluded to in the first paragraph of § 1, the conclusion of Corollaries 1 and 2 is not that the origin is asymptotically stable in the usual sense, see [*3*, p. 142], but merely that all solutions of (1.1) tend to the origin as $t \to \infty$. For the origin also to be stable, it is sufficient to add

$$p(t, 0, 0) \equiv 0, \quad q(t, 0, 0) \equiv 0, \quad e_1(t) = e_2(t) \equiv 0 \qquad (t \in I) \tag{5.3}$$

to the hypothesis of Lemma 3.1, as is evident from (3.3). Invoking (5.3) rules out, of course, many of the problems that are included in the present considerations, *e.g.* (1.4) with $e(t) \not\equiv 0$. For the origin also to be uniformly stable, it is sufficient to assume further that

$$\lim_{|x| + |y| \to 0} E(t, x, y) = 0 \quad \text{uniformly on } t \in I.$$

Added in Proof. Recently J. P. La Salle in the Proc. N. A. S. **46**, 363–365 (1960), has announced results concerning autonomous systems which overlap considerably those of the present paper when the latter are restricted to (1.6).

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On Approximate Solutions of Non-linear Hyperbolic Partial Differential Equations

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§ 1. Introduction

Consider the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1.1}$$

given together with the characteristic initial condition

$$u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y) \quad \bigl(\text{where } \sigma(0) = \tau(0)\bigr) \tag{1.2}$$

to be satisfied by $u(x, y)$ on the rectangle $\mathscr{R}$: $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

In comparison, consider also the ordinary differential equation

$$u' = f(x, u) \tag{1.3}$$

with the initial condition

$$u(0) = \sigma \quad (= \text{constant}) \tag{1.4}$$

to be satisfied by $u(x)$ on the interval $\mathscr{I}$: $0 \leqq x \leqq a$.

These two problems will be denoted respectively by 2-HP and 1-Ord.

Many authors have realized the analogy between these initial value problems in two and one independent variables. The object of this paper is to develop a proof of the existence (under suitable conditions) of a solution of 2-HP which is analogous in some respects to the existence proof given by PEANO [9] for 1-Ord. The important part of this development is that a certain broad class of "approximate solution functions" are indeed approximations to a solution in the sense that, under a natural ordering, they converge uniformly to a solution. The actual conclusion of the existence of a solution under the given conditions is less important, for this is already known (*cf.* HARTMAN & WINTNER [5], P. LEEHEY [7] and J. B. DIAZ [3]).

A subsidiary aim, for the purposes of the present paper, will be to extract the essential notions in the development of the proof in such a manner as to suggest how a proof might be developed for analogous equations of higher order in correspondingly more independent variables.

In a subsequent paper, an analog of the Runge-Kutta procedure for numerically obtaining approximate solutions of 1-Ord. will be developed for 2-HP. The procedure derived there is similar both in application and order of accuracy to

that given by KUTTA [6]. Its validity depends on the proof in the present paper; indeed, the numerical process defines an approximate solution which is included in the class of approximate solutions investigated here.

In viewing 2-HP as an analog of 1-Ord., one may consider generalizing the Euler-Cauchy polygon method of finding approximate solutions of 1-Ord. In this method, one subdivides the interval $\mathscr{I}$ at the points $0=\xi_0, \xi_1, \ldots, \xi_m=a$ into subintervals $\Delta\xi_i=\xi_{i+1}-\xi_i$ and defines the function $v(x)$ on $\mathscr{I}$ by

$$(1.5)\qquad v(x)=\sigma+\sum_{\alpha=0}^{i-1} C_\alpha\,\Delta\xi_\alpha+C_i(x-\xi_i),\quad \text{where}\quad C_\alpha=f\big(\xi_\alpha, v(\xi_\alpha)\big),$$

for $\xi_i\leqq x<\xi_{i+1}$. This defines $v(x)$ as a continuous polygon (*i.e.* piecewise linear function) through $(0,\sigma)$ with vertices where $x=\xi_i$. ZWIRNER [10] gave a generalization of this for the special case of 2-HP when $u_{xy}=f(x, y, u)$. This he did by subdividing $\mathscr{R}$ into subrectangles in the usual manner by a mesh of lines, and defining over each subrectangle a bilinear function (hyperbolic paraboloid) in such a manner that these bilinear functions could be pieced together to obtain a continuous function approximating a solution on all of $\mathscr{R}$. Analytically, such an approximation can be formulated by choosing division lines $x=\xi_i$ and $y=\eta_j$ in $\mathscr{R}$, letting $\Delta\xi_i=\xi_{i+1}-\xi_i$, $\Delta\eta_j=\eta_{j+1}-\eta_j$ and $\bar\sigma(x)=\sigma(\xi_i)+\dfrac{\sigma(\xi_{i+1})-\sigma(\xi_i)}{\Delta\xi_i}(x-\xi_i)$ for $\xi_i\leqq x<\xi_{i+1}$, $\bar\tau(y)=\tau(\eta_j)+\dfrac{\tau(\eta_{j+1})-\tau(\eta_j)}{\Delta\eta_j}(y-\eta_j)$ for $\eta_j\leqq y<\eta_{j+1}$ and finally setting

$$(1.6)\qquad \begin{aligned} v(x, y)&=\bar\sigma(x)+\bar\tau(y)-\sigma(0)+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-1} C_{\alpha\beta}\,\Delta\xi_\alpha\,\Delta\eta_\beta+\\ &\quad+\sum_{\beta=0}^{j-1} C_{i\beta}\,\Delta\eta_\beta\,(x-\xi_i)+\sum_{\alpha=0}^{i-1} C_{\alpha j}\,\Delta\xi_\alpha\,(y-\eta_j)+C_{ij}(x-\xi_i)(y-\eta_j),\end{aligned}$$

where $C_{\alpha\beta}$ is a constant approximately equal to $f\big(\xi_\alpha, \eta_\beta, v(\xi_\alpha, \eta_\beta)\big)$. If exact equality were required here for $C_{\alpha\beta}$, then the analog with the polygon method for 1-Ord. would be precise; but in fact this greater degree of generality is permitted, the only requirement being that as the mesh is made finer the value $C_{\alpha\beta}$ should tend to $f\big(\xi_\alpha, \eta_\beta, v(\xi_\alpha, \eta_\beta)\big)$. Actually ZWIRNER gives specific restrictions on how $C_{\alpha\beta}$ may deviate from f, but the requirement mentioned is satisfied because of these restrictions.

Both (1.5) and (1.6) can be motivated by putting the corresponding differential equations into integral form and then carrying out an approximate integration with a piecewise constant integrand. Putting (1.1) and (1.2) into integral form, one finds

$$(1.7)\qquad \begin{aligned} u(x, y)&=\sigma(x)+\tau(y)-\sigma(0)+\int_0^x\!\!\int_0^y f\big(s, t, u(s,t), p(s,t), q(s,t)\big)\,ds\,dt,\\ p(x, y)&=\sigma'(x)+\int_0^y f\big(x, t, u(x,t), p(x,t), q(x,t)\big)\,dt,\\ q(x, y)&=\tau'(y)+\int_0^x f\big(s, y, u(s,y), p(s,y), q(s,y)\big)\,ds.\end{aligned}$$

These form the starting point of the analog of the Euler-Cauchy method developed recently by DIAZ [3]. The integrals are replaced by summations quite similar

to those in (1.5) and (1.6). In particular, (1.6) yields an approximate solution of (1.1) by taking (*cf.* p. 368 of DIAZ's paper)

$$C_{\alpha\beta} = f\left(\xi_\alpha, \eta_\beta, v(\xi_\alpha, \eta_\beta), \frac{v(\xi_{\alpha+1}, \eta_\beta) - v(\xi_\alpha, \eta_\beta)}{\xi_{\alpha+1} - \xi_\alpha}, \frac{v(\xi_\alpha, \eta_{\beta+1}) - v(\xi_\alpha, \eta_\beta)}{\eta_{\beta+1} - \eta_\beta}\right).$$

This choice of $C_{\alpha\beta}$ yields approximating functions for (1.1) quite analogous to the Euler-Cauchy polygons for (1.3). Again the approximating surface is made up of segments of hyperbolic paraboloids.

By including u_x and u_y among the arguments of f, one encounters an important difficulty. When in the second and third of equations (1.7) the integrand f is replaced by a piecewise constant function C, the resulting approximations to p and q are not continuous. For the approximations to p, for example, there is no integration with respect to x to smooth out the discontinuities of C on lines $x = \xi_\alpha$. This difficulty was encountered by DIAZ. To overcome it, he derived a lemma (stated in § 3 below) which extracts the needed properties. On the strength of its conclusion one may obtain the needed convergence of the approximations of the derivatives by applying a theorem of ARZELÀ which deals with the convergence of certain not necessarily continuous functions to a continuous limit function. J. CONLAN [2] gives a proof of ARZELÀ's theorem in an appendix to a paper in which he extends DIAZ's method to the Cauchy and Mixed Boundary-Value Problems.

In the development below for 2-HP, the domain space of independent variables will be divided as usual into subrectangles. Now it may be observed in both the Euler-Cauchy polygons for 1-Ord. and their analogs for 2-HP that the most important values of the approximate solutions are those on the nodes of the meshes. The values at intermediate points are unimportant. For example, for 1-Ord. the polygonal approximation (1.5) could be replaced by a step function agreeing with the polygon at the left endpoint of each subinterval of I. The sequence of step functions so obtained still converges uniformly to the solution. Therefore, so far as existence proofs and numerical processes are concerned, the values at non-node points as given by a continuous polygon are of no concern. In this presentation we have thus omitted any consideration at all of the intermediate values when defining the functions approximating the solutions. Upon defining these functions in the usual stepwise manner beginning with the initial conditions, one may observe further that they satisfy certain "growth conditions" which appear in the form of summations resulting from approximate integration of the integral equations equivalent to the given differential equation. These growth properties lead, via the inequality in DIAZ's lemma, to the satisfying of the "smallness of discontinuity" requirement of ARZELÀ's theorem, which in turn assures the existence of the desired subsequence of approximate solutions converging uniformly to a solution.

In accordance with the above, in § 2 ARZELÀ's theorem is given in a generalized form for functions whose domains are the node sets of convergent meshes in Euclidean n-space and whose ranges are in a complete metric space. In § 3 the concept of a "vine" is introduced, a vine being a function defined on mesh nodes whose image at a given node depends only on the images at preceding nodes. The basic result of the section (Theorem 2) shows that for sequences of vines

defined on convergent meshes and satisfying certain growth conditions there are subsequences converging uniformly to continuous functions. In § 4 it is shown through two lemmas that sequences of vines of the type considered in Theorem 2 and appropriately related to the problem 2-HP do exist, and that the limit functions so determined yield solutions to the problem. These results are summarized in Theorem 3, which is our principal result. The final theorem, given in § 5, shows that every continuously differentiable solution of 2-HP is the limit of a sequence of approximating functions of the type under consideration. This answers affirmatively, for $u_{xy}=f(x, y, u, p, q)$, a question suggested by the results of ZWIRNER [*10*] concerning the equation $u_{xy}=f(x, y, u)$.

§ 2. Functions on Mesh Nodes

Throughout the following i will have the range $1, \ldots, n$.

In n-dimensional real space let

1. x denote the point $(x^1, x^2, \ldots, x^n)$;
2. $\|x\| = \sum_{i=1}^{n} |x^i|$;
3. $\mathscr{X}$ denote the cell $0 \leq x^i \leq a^i$ for some fixed values a^i;
4. $\{\mathscr{M}^\nu\}$ be a sequence of meshes on $\mathscr{X}$ formed by the hyperplanes

$$x^i = \xi_k^{i\nu} \qquad (k=0, 1, \ldots, m^{i\nu}) \qquad (\text{where } \xi_0^{i\nu}=0 \text{ and } \xi_{m^{i\nu}}^{i\nu}=a^i)$$

chosen such that $\Delta_k^{i\nu} = \xi_{k+1}^{i\nu} - \xi_k^{i\nu}$ is positive;

5. $\mathscr{N}^\nu$ denote the set of nodes of $\mathscr{M}^\nu$, that is $\mathscr{N}^\nu = \{(\xi_{k_1}^{1\nu}, \xi_{k_2}^{2\nu}, \ldots, \xi_{k_n}^{n\nu})\}$;
6. $\xi_{k_1 k_2 \ldots k_n}^\nu$ denote the node $(\xi_{k_1}^{1\nu}, \xi_{k_2}^{2\nu}, \ldots, \xi_{k_n}^{n\nu})$;
7. $\|\mathscr{N}^\nu\| = \max_{k_1, \ldots, k_n} \|\xi_{k_1+1, k_2+1, \ldots, k_n+1}^\nu - \xi_{k_1 k_2 \ldots k_n}^\nu\|$.

Thus, if $\lim_\nu \|\mathscr{N}^\nu\| = 0$, then the diameter of the largest subcell of $\mathscr{M}^\nu$ approaches zero as $\nu \to \infty$, and the points in $\bigcup_\nu \mathscr{N}^\nu$ are dense in $\mathscr{X}$.

Definition. A sequence of meshes $\{\mathscr{M}^\nu\}$ such that $\lim_\nu \|\mathscr{N}^\nu\| = 0$ will be called *convergent.*

Let $\mathscr{V}$ be a complete metric space with distance function ϱ. For a given sequence $\{\mathscr{M}^\nu\}$ of meshes in $\mathscr{X}$ let $\{V^\nu\}$ be a corresponding sequence of functions $V^\nu\colon \mathscr{N}^\nu \to \mathscr{V}$.

Definition. A sequence $\{V^\nu\}$ of functions $V^\nu\colon \mathscr{N}^\nu \to \mathscr{V}$ will be called *Arzelà quasi-continuous*, or an *Arzelà sequence*, if and only if given $\varepsilon > 0$ there exist δ_ε and N_ε such that $\nu > N_\varepsilon$ and

$$\|\xi_{k_1 k_2 \ldots k_n}^\nu - \xi_{k_1 k_2' \ldots k_n'}^\nu\| < \delta_\varepsilon$$

imply

$$\varrho\left(V^\nu(\xi_{k_1 \ldots k_n}^\nu), V^\nu(\xi_{k_1 \ldots k_n'}^\nu)\right) < \varepsilon.$$

Note that this property of sequences is less restrictive than equi-continuity in that, even when an analogous definition is made for functions $\mathscr{X} \to \mathscr{V}$, the functions need not be continuous. It does require, however, that any discontinuities become small as $\nu \to \infty$.

Definition. The functions V^ν will be said to *converge uniformly* to a function V of $\mathscr{X}$ into $\mathscr{V}$ if given $\varepsilon > 0$ there is an N_ε such that $\nu > N_\varepsilon$ implies $\varrho\left(V^\nu(\xi^\nu_{k_1\ldots k_n}), V(\xi^\nu_{k_1\ldots k_n})\right) < \varepsilon$ for each $\xi^\nu_{k_1\ldots k_n} \in \mathscr{N}^\nu$.

Although the functions V^ν are defined only on $\mathscr{N}^\nu$, there is a simple extension to all of $\mathscr{X}$: If $\xi^{i\nu}_{k_i} \leqq x^i < \xi^{i\nu}_{k_i+1}$ for each i (this includes the possibility $x^i = a^i$), then let $V^{+\nu}(x) = V^\nu(\xi^\nu_{k_1\ldots k_n})$. Note that the uniform convergence (in the usual sense) of $V^{+\nu}$ to V implies the uniform convergence (in the above sense) of V^ν to V. This extension is useful in the proof of the generalization of ARZELÀ's theorem (Theorem 1 below), but it is not otherwise needed in the sequel.

In case $\mathscr{X}$ is simply a rectangle in E^2 and $\mathscr{V}$ is a linear space, another way to extend the definition of $\mathscr{V}^\nu$ to all of $\mathscr{X}$ is bilinearly over each subrectangle (*cf.* DIAZ [3]).

Theorem 1 (ARZELÀ). *Let $\mathscr{X}$ be a cell in E^n, and let $\{\mathscr{M}^\nu\}$ be a convergent sequence of meshes in $\mathscr{X}$. Let $\mathscr{N}^\nu$ denote the nodes of $\mathscr{M}^\nu$. Let $\mathscr{V}$ be a complete metric space and $\{V^\nu\}$ an Arzelà quasi-continuous sequence of functions $V^\nu: \mathscr{N}^\nu \to \mathscr{V}$ such that the ranges of all the V^ν lie in a compact subset of $\mathscr{V}$.*

Then there is a subsequence $\{V^{\nu_j}\}$ of $\{V^\nu\}$ and a continuous function $V: \mathscr{X} \to \mathscr{V}$ such that V^{ν_j} converges uniformly to V as $j \to \infty$.

This theorem is easily proved by first employing the extended functions $V^{+\nu}$ above mapping $\mathscr{X}$ into $\mathscr{V}$. For the sequence $\{V^{+\nu}\}$ the proof follows in a fashion almost identical with that for a sequence of equicontinuous functions, namely, via the diagonal process employed in connection with the countable dense subset of $\mathscr{X}$ given by the separability. Having found in this manner a subsequence $\{V^{+\nu_j}\}$ converging to a function V, one employs the corresponding subsequence $\{V^{\nu_j}\}$ and the same function V to prove the theorem. (See CONLAN [2], Appendix 2, for a complete proof for the case of functions defined on the whole of any compact separable metric space $\mathscr{X}$.)

Note that this theorem provides a continuous function on all of $\mathscr{X}$ as the limit of a sequence of functions each defined on discrete point sets. It serves thus as a bridge from functions with discrete domain to functions with continuous domain. In numerical work, it is the former which are actually employed as approximate solutions of differential equations. In succeeding sections it is functions of this type which are studied. At the end, Theorem 1 is used to provide a continuous function which is a solution of 2-HP.

§ 3. Vines

In this section we consider specific functions V^ν defined on $\mathscr{N}^\nu$ which are suggested by the observations in the Introduction.

Let $\mathscr{N}^\nu$ be partially ordered in the natural manner: $\xi^\nu_{k_1\ldots k_n} \leqq \xi^\nu_{k'_1\ldots k'_n}$ provided $k_i \leqq k'_i$ $(i = 1, \ldots, n)$. The symbol $<$ will mean both $\leqq$ and $\neq$ hold.

Definition. Let V^ν be as above, a mapping of $\mathscr{N}^\nu$ into $\mathscr{V}$. V^ν will be called a *vine* if the value $V^\nu(\xi^\nu_{k_1\ldots k_n})$ depends on the values $V^\nu(\xi^\nu_{k'_1\ldots k'_n})$ for $\xi^\nu_{k'_1\ldots k'_n} < \xi^\nu_{k_1\ldots k_n}$ but is known as soon as these values $V^\nu(\xi^\nu_{k'_1\ldots k'_n})$ are known.

For notational convenience let $V^\nu_{k_1\ldots k_n} = V^\nu(\xi^\nu_{k_1\ldots k_n})$.

The conditions on V^ν now to be considered are given for $\mathscr{X}$ in 2-dimensional real cartesian space R^2; they are motivated by the second and third of the integral

equations (1.7). The statements following are thus designed to apply to the special case 2-HP of present interest. However, it is hoped that this interpretation is suggestive for higher dimensional cases.

Suppose, for some constant $L \geqq 0$ and some quantities $\varepsilon_{kk'}$ and $\varepsilon_{ll'} \geqq 0$, that the vines V^ν satisfy, for $\nu > N_G$ and $\|\xi^\nu_{kl} - \xi^\nu_{k'l'}\| < \delta_G$, the inequalities:

$$G_1\colon\quad \varrho(V^\nu_{kl}, V^\nu_{k'l}) \leqq \varepsilon_{kk'} + L\sum_{j=0}^{l-1} \varrho(V^\nu_{kj}, V^\nu_{k'j})\, \Delta^{2\nu}_j, \qquad l = 0, 1, \ldots, m^{2\nu};$$

$$G_2\colon\quad \varrho(V^\nu_{kl}, V^\nu_{kl'}) \leqq \varepsilon_{ll'} + L\sum_{i=0}^{k-1} \varrho(V^\nu_{il}, V^\nu_{il'})\, \Delta^{1\nu}_i, \qquad k = 0, 1, \ldots, m^{1\nu};$$

where, by convention, sums $\sum\limits_{0}^{-1}$ are zero. The vines V^ν will then be said to satisfy the *growth conditions* G_1 and G_2. The quantities $\varepsilon_{kk'}$ and $\varepsilon_{ll'}$ can be thought of as *initial oscillation bounds* of V^ν. Such quantities, in the application to 2-HP, will be related to the initial conditions σ and τ.

With the preceding growth conditions as background, a lemma due to DIAZ ([3], p. 373) will now be stated. This lemma is the key to proving that the sequence of vines is Arzelà quasi-continuous and subsequently that it converges to a continuous function V of $\mathscr{X}$ into $\mathscr{V}$.

Lemma 1. *If*

(a) *the interval* $[0, A]$ *is subdivided into finitely many subintervals* $\Delta_0, \Delta_1, \ldots, \Delta_{t-1}$;

(b) $g_0, g_1, \ldots, g_t$ *is a set of non-negative real numbers*;

(c) *the numbers* $L \geqq 0$ *and* $\varepsilon \geqq 0$ *are such that the inequality*

$$g_s \leqq \varepsilon + L\sum_{r=0}^{s-1} g_r\, \Delta_r$$

holds for $s = 1, \ldots, t$;

then the inequality

$$g_s \leqq \left\{\prod_{r=0}^{s-1} [1 + L\,\Delta_r]\right\} \{\varepsilon + L\, g_0\, \Delta_0\} \leqq e^{LA} \{\varepsilon + L\, g_0\, \Delta_0\}$$

holds for $s = 1, \ldots, t$.

Remark. If, moreover, $g_0 < \varepsilon$, then $g_s < \varepsilon e^{LA}(1 + L\Delta_0)$ for $s = 1, 2, \ldots, t$. This is an immediate consequence of the lemma.

The last result applies immediately to vines V^ν satisfying the growth conditions G_1 and G_2. By setting $[0, A] = [0, a^2]$, $\Delta_j = \Delta^{2\nu}_j$, $\varepsilon = \varepsilon_{kk'}$ and $g_l = \varrho(V^\nu_{kl}, V^\nu_{k'l})$, it follows from G_1 and the lemma that, for $\nu > N_G$ and $\|\xi^\nu_{kl} - \xi^\nu_{k'l'}\| < \delta_G$,

$$\varrho(V^\nu_{kl}, V^\nu_{k'l}) \leqq \varepsilon_{kk'}\, e^{La^2}(1 + L\,\Delta^{2\nu}_0).$$

Similarly, from G_2 it follows that

$$\varrho(V^\nu_{kl}, V^\nu_{kl'}) \leqq \varepsilon_{ll'}\, e^{La^1}(1 + L\,\Delta^{1\nu}_0).$$

If the sequence of meshes $\{\mathscr{M}^\nu\}$ on which the V^ν are defined is convergent, then there is an N_1 such that $\nu > N_1$ implies $L\Delta^{1\nu}_0 < 1$ and $L\Delta^{2\nu}_0 < 1$. Thus, for

$\nu > \max(N_1, N_G)$, $\quad \varrho(V_{kl}^\nu, V_{k'l'}^\nu) \leqq 2(\varepsilon_{kk'} e^{La^2} + \varepsilon_{ll'} e^{La^1})$.

Suppose further that $\varepsilon_{kk'}$ and $\varepsilon_{ll'}$ are such that given $\varepsilon > 0$ there exist δ_2 and N_2 such that the conditions $\nu > N_2$ and $\|\xi_{kl}^\nu - \xi_{k'l'}^\nu\| < \delta_2$ imply $2(\varepsilon_{kk'} e^{La^2} + \varepsilon_{ll'} e^{La^1}) < \varepsilon$ on all of $\mathscr{N}^\nu$. Then from $\nu > N_\varepsilon = \max(N_G, N_1, N_2)$ and $\|\xi_{kl}^\nu - \xi_{k'l'}^\nu\| < \delta_\varepsilon = \min(\delta_G, \delta_2)$ it follows that

$$\varrho(V_{kl}^\nu, V_{k'l'}^\nu) < \varepsilon;$$

that is, the sequence $\{\mathscr{V}^\nu\}$ is an Arzelà sequence, and the Arzelà theorem immediately applies. The foregoing results may be collected into the following theorem.

Theorem 2. *Let $\mathscr{X}$ be a cell $\{(x^1, x^2): 0 \leq x^i \leqq a^i\}$ in 2-dimensional real space; let $\mathscr{V}$ be a complete metric space. If*

(a) *$\{\mathscr{M}^\nu\}$ is a convergent sequence of meshes on $\mathscr{X}$;*

(b) *$\{V^\nu\}$ is a sequence of vines, $V^\nu: \mathscr{N}^\nu \to \mathscr{V}$;*

(c) *the V^ν satisfy the growth conditions G_1 and G_2;*

(d) *the initial oscillation bounds $\varepsilon_{kk'}$ and $\varepsilon_{ll'}$ are such that given $\varepsilon > 0$ there is an N_ε and a δ_ε such that $\nu > N_\varepsilon$ and $\|\xi_{kl}^\nu - \xi_{k'l'}^\nu\| < \delta_\varepsilon$ imply $\varepsilon_{kk'} + \varepsilon_{ll'} < \varepsilon$;*

then there is a subsequence $\{V^{\nu_j}\}$ and a continuous function V of $\mathscr{X}$ into $\mathscr{V}$ such that $\{V^{\nu_j}\}$ converges uniformly to V as $j \to \infty$.

§ 4. Approximate Solutions, and their Convergence, for 2-HP

In the application of the preceding to differential equations, satisfaction of the growth conditions depends on a function $f = f(x, v)$ of $\mathscr{X} \times \mathscr{V} \to R$, the real field. In the present case this is the function f appearing in equation (1.1). In order to develop the relationship between vines and the problem 2-HP let:

1. $\mathscr{X}$ denote the cell $0 \leqq x^i \leqq a^i$ $(i = 1, 2)$ in 2-dimensional real space, and let the notation of § 3 apply to $\mathscr{X}$;
2. $\mathscr{V}$ be 3-dimensional real space with points $v = (v^0, v^1, v^2)$;
3. $\|v\| = \sum_{j=0}^{2} |v^j|$;
4. $\{\mathscr{M}_{\mathscr{V}}^\nu\}$ be a sequence of (infinite) meshes on $\mathscr{V}$ formed by the planes

$$v^j = \omega_r^{j\nu} \qquad (r = 0, \pm 1, \pm 2, \ldots)$$

chosen such that $\omega_{r+1}^{j\nu} - \omega_r^{j\nu}$ is positive;

5. $\omega_{r_0 r_1 r_2}^\nu$ denote the point $(\omega_{r_0}^{0\nu}, \omega_{r_1}^{1\nu}, \omega_{r_2}^{2\nu})$;
6. $\|\mathscr{M}_{\mathscr{V}}^\nu\| = \limsup_{r_0, r_1, r_2} \|\omega_{r_0+1, r_1+1, r_2+1}^\nu - \omega_{r_0 r_1 r_2}^\nu\|$ when this exists;
7. $\mathscr{M}^{*\nu}$ be the mesh on $\mathscr{X} \times \mathscr{V}$ formed by $\mathscr{M}^\nu$ and $\mathscr{M}_{\mathscr{V}}^\nu$ together;
8. $\|\mathscr{M}^{*\nu}\| = \max\left(\|\mathscr{N}^\nu\|, \|\mathscr{M}_{\mathscr{V}}^\nu\|\right)$ when this exists.

As with $\mathscr{M}^\nu$, the sequence of meshes $\{\mathscr{M}^{*\nu}\}$ will be called *convergent* if $\|\mathscr{M}^{*\nu}\| \to 0$ as $\nu \to \infty$. The sets $\{(x, v)\}$ of $\mathscr{X} \times \mathscr{V}$ such that $\xi_{k_i}^{i\nu} \leq x^i < \xi_{k_i+1}^{i\nu}$, $\omega_{r_j}^{j\nu} \leq v^j < \omega_{r_j+1}^{j\nu}$ will be called *cells* of $\mathscr{M}^{*\nu}$.

If $V^{0\nu}$, $V^{1\nu}$ and $V^{2\nu}$ are real valued functions on $\mathscr{N}^\nu$, let

$$V_{kl}^{j\nu} = V^{j\nu}(\xi_{kl}^\nu) = V^{j\nu}(\xi_k^{1\nu}, \xi_l^{2\nu}) \quad \text{and} \quad V_{kl}^\nu = (V_{kl}^{0\nu}, V_{kl}^{1\nu}, V_{kl}^{2\nu})$$

so that V^ν is a function on $\mathscr{N}^\nu$ into $\mathscr{V}$. (In the following, V^ν will be a vine mapping $\mathscr{N}^\nu$ into $\mathscr{V}$.)

For each ν, let $\mathscr{C}^\nu$ be a real-valued function, $\mathscr{C}^\nu(x, v) = \mathscr{C}^\nu(x^1, x^2, v^0, v^1, v^2)$ defined on $\mathscr{X} \times \mathscr{V}$ such that $\mathscr{C}^\nu$ is constant on each cell of $\mathscr{M}^{*\nu}$. Here, as later, a point in $\mathscr{X} \times \mathscr{V}$ will be denoted by (x, v) or $(x^1, x^2, v^0, v^1, v^2)$ as notational convenience dictates.

Lastly, let $C^\nu_{kl} = C^\nu(\xi^\nu_{kl}) = \mathscr{C}^\nu(\xi^\nu_{kl}, V^\nu_{kl})$.

Lemma 2. *(Existence of vines satisfying the growth conditions.) Suppose $\sigma(x^1)$ and $\tau(x^2)$ are continuously differentiable on the intervals $[0, a^1]$ and $[0, a^2]$, respectively, and $\sigma(0) = \tau(0)$. Suppose f is a real-valued function $f(x, v) = f(x^1, x^2, v^0, v^1, v^2)$ defined on $\mathscr{X} \times \mathscr{V}$ which is continuous and bounded and satisfies on $\mathscr{X} \times \mathscr{V}$ the Lipschitz condition with Lipschitz constant L:*

$$|f(x^1, x^2, v^0, v^1, v^2) - f(x^1, x^2, v^0, \bar{v}^1, \bar{v}^2)| \leqq L\{|v^1 - \bar{v}^1| + |v^2 - \bar{v}^2|\}.$$

Let $\mathscr{M}^\nu$ and $\mathscr{M}^{*\nu}$ be convergent meshes on $\mathscr{X}$ and $\mathscr{X} \times \mathscr{V}$, respectively. Let $\mathscr{C}^\nu$ and C^ν be as above, and suppose further that $\mathscr{C}^\nu$ converges uniformly to f as $\nu \to \infty$.

Then there is a sequence of vines V^ν of $\mathscr{N}^\nu$ into $\mathscr{V}$ satisfying the hypotheses of Theorem 2.

Note. In a later use of the lemma, three sequences $\{\mathscr{C}^{j\nu}\}$ $(j = 0, 1, 2)$ will be used to define respectively the $V^{0\nu}$, $V^{1\nu}$, $V^{2\nu}$ appearing in the proof below. See the remark preceding Theorem 3, § 4.

Proof. Let $\sigma^\nu_k = \sigma(\xi^{1\nu}_k)$ and $\tau^\nu_l = \tau(\xi^{2\nu}_l)$. Define σ'^ν_k and τ'^ν_l similarly. For each ν define

$$\begin{aligned} V^{0\nu}_{kl} &= \sigma^\nu_k + \tau^\nu_l - \sigma_0 + \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} C^\nu_{\alpha\beta} \Delta^{1\nu}_\alpha \Delta^{2\nu}_\beta \\ V^{1\nu}_{kl} &= \sigma'^\nu_k + \sum_{\beta=0}^{l-1} C^\nu_{k\beta} \Delta^{2\nu}_\beta \\ V^{2\nu}_{kl} &= \tau'^\nu_l + \sum_{\alpha=0}^{k-1} C^\nu_{\alpha l} \Delta^{1\nu}_\alpha \end{aligned} \tag{4.1}$$

where by convention, $\sum_0^{-1} = 0$. Set $V^\nu = (V^{0\nu}, V^{1\nu}, V^{2\nu})$. The sequence $\{V^\nu\}$ will be shown to be as desired.

For notational simplicity, the index ν will be suppressed in the remainder of the proof, but, for later use, given ε_C, will be thought of as so large, say $\nu > N_C$, that $|\mathscr{C}^\nu(x, v) - f(x, v)| < \varepsilon_C$.

It is easily checked that V is well defined. Furthermore, V is a vine.

Hypotheses (a) and (b) of Theorem 2 are thus satisfied; it will be shown that (c) and (d) are also satisfied.

First, note that V is bounded. For, since f is bounded, $\mathscr{C}$ is also bounded, say by M, while, because σ, σ', τ and τ' are continuous on compact sets, they too are bounded, say by the same M. Thus, it is clear from the defining equations (4.1) for V that

$$|V^0| \leqq 3M + M a^1 a^2, \qquad |V^1| \leqq M + M a^2 \quad \text{and} \quad |V^2| \leqq M + M a^1.$$

Hence, f is considered only on a bounded portion of $\mathscr{X}\times\mathscr{V}$, and, being continuous on all of $\mathscr{X}\times\mathscr{V}$, is thus uniformly continuous in the domain of interest. Also, σ, σ', τ and τ' are uniformly continuous on $[0, a^1]$ and $[0, a^2]$, respectively.

Next,

$$\|V_{kl}-V_{k'l}\|=\sum_{j=0}^{2}|V_{kl}^j-V_{k'l}^j|, \tag{4.2}$$

in which, with $k^*=\max(k, k')$ and $k_*=\min(k, k')$, one has from (4.1),

$$|V_{kl}^0-V_{k'l}^0|\leqq|\sigma_k-\sigma_{k'}|+\sum_{\alpha=k_*}^{k^*-1}\sum_{\beta=0}^{l-1}|C_{\alpha\beta}|\,\Delta_\alpha^1\Delta_\beta^2,$$

$$|V_{kl}^1-V_{k'l}^1|\leqq|\sigma_k'-\sigma_{k'}'|+\sum_{\beta=0}^{l-1}|C_{k\beta}-C_{k'\beta}|\,\Delta_\beta^2,$$

$$|V_{kl}^2-V_{k'l}^2|\leqq\sum_{\alpha=k_*}^{k^*-1}|C_{\alpha l}|\,\Delta_\alpha^1.$$

Because $|\mathscr{C}|\leqq M$ and σ is uniformly continuous, it is clear that for $l=0, 1, \ldots, m^{2\nu}$ both $|V_{kl}^0-V_{k'l}^0|$ and $|V_{kl}^2-V_{k'l}^2|$ can be made less than a given e_0 by taking $|\xi_k^1-\xi_{k'}^1|$ less than some d_0.

The difference $|V_{kl}^1-V_{k'l}^1|$ is not so simply shown to be small. Indeed, this difference yields the summation which appears in the growth condition G_1. To begin with, $|C_{k\beta}-C_{k'\beta}|=|\mathscr{C}(\xi_{k\beta}, V_{k\beta})-\mathscr{C}(\xi_{k'\beta}, V_{k'\beta})|$, so that, since $\nu>N_C$,

$$\begin{aligned}|C_{k\beta}-C_{k'\beta}|&\leqq|f(\xi_{k\beta}, V_{k\beta})-f(\xi_{k'\beta}, V_{k'\beta})|+2\varepsilon_C\\ &\leqq|f(\xi_k^1, \xi_\beta^2, V_{k\beta}^0, V_{k\beta}^1, V_{k\beta}^2)-f(\xi_{k'}^1, \xi_\beta^2, V_{k\beta}^0, V_{k\beta}^1, V_{k\beta}^2)|\\ &+|f(\xi_{k'}^1, \xi_\beta^2, V_{k\beta}^0, V_{k\beta}^1, V_{k\beta}^2)-f(\xi_{k'}^1, \xi_\beta^2, V_{k'\beta}^0, V_{k\beta}^1, V_{k\beta}^2)|\\ &+|f(\xi_{k'}^1, \xi_\beta^2, V_{k'\beta}^0, V_{k\beta}^1, V_{k\beta}^2)-f(\xi_{k'}^1, \xi_\beta^2, V_{k'\beta}^0, V_{k'\beta}^1, V_{k\beta}^2)|\\ &+|f(\xi_{k'}^1, \xi_\beta^2, V_{k'\beta}^0, V_{k'\beta}^1, V_{k\beta}^2)-f(\xi_{k'}^1, \xi_\beta^2, V_{k'\beta}^0, V_{k'\beta}^1, V_{k'\beta}^2)|+2\varepsilon_C\end{aligned}$$

or

$$|C_{k\beta}-C_{k'\beta}|\leqq A_1+A_2+A_3+A_4+2\varepsilon_C,$$

where in the last inequality the A_i are the respective terms in the previous sum. A bound will now be found for each A_i.

From the comment following (4.2), and the uniform continuity of f, each of A_1, A_2 and A_4 can be made less than $\frac{1}{3}\varepsilon_1$ by taking $|\xi_k^1-\xi_{k'}^1|<d_1$.

By the Lipschitz property of f,

$$A_3<L|V_{k\beta}^1-V_{k'\beta}^1|,$$

and thus one finds that

$$|C_{k\beta}-C_{k'\beta}|\leqq\varepsilon_1+2\varepsilon_C+L|V_{k\beta}^1-V_{k'\beta}^1|. \tag{4.3}$$

Hence, using also the continuity of σ', one finds from (4.2) that, given $\varepsilon>0$,

$$\|V_{kl}-V_{k'l}\|\leqq e+(\varepsilon_1+2\varepsilon_C)\,a^2+L\sum_{\beta=0}^{l-1}|V_{k\beta}^1-V_{k'\beta}^1|\,\Delta_\beta^2,$$

provided that $|\xi_k^1-\xi_{k'}^1|$ is less than an appropriate δ_1 and $\nu>N_C$. Because of the *uniform* continuity of f, σ, σ', τ and τ', this holds throughout $\mathscr{N}^\nu$.

Letting $\varepsilon_{kk'} = e + (\varepsilon_1 + 2\varepsilon_C)\, a^2$, one has

$$\| V_{kl} - V_{k'l} \| \leqq \varepsilon_{kk'} + L \sum_{\beta=0}^{l-1} \| V_{k\beta} - V_{k'\beta} \| \Delta_\beta^2$$

so that V satisfies the growth condition G_1.

Similarly, for suitable δ_2 and N_C, if $|\xi_l^2 - \xi_{l'}^2| < \delta_2$ and $\nu > N_C$, then $\| V_{kl} - V_{kl'} \| \leqq \varepsilon_{ll'} + L \sum_{\alpha=0}^{k-1} \| V_{\alpha l} - V_{\alpha l'} \| \Delta_\alpha^1$ so that V satisfies the growth condition G_2.

Further, the hypothesis (d) of Theorem 2 does hold because of the *uniform* continuity of f, σ, σ', τ and τ'. This completes the proof of the lemma.

Theorem 2 can now be applied to the sequence of vines $\{V^\nu\}$ supplied by Lemma 2. Thus, there is a subsequence $\{V^{\nu_j}\}$ of $\{V^\nu\}$ which converges uniformly as $j \to \infty$ to a *continuous* function $V: \mathscr{X} \to \mathscr{V}$. Because $\mathscr{V}$ is (here) three-dimensional real space, the function V defines three real-valued functions V^0, V^1 and V^2, the components of V, which are again continuous and are the uniform limits of $V^{0\nu}$, $V^{1\nu}$ and $V^{2\nu}$, respectively. The next lemma asserts V^0 is a solution of 2-HP.

For this remaining lemma, the more common (x, y)-notation for two independent variables will be used since it is somewhat more convenient for the proof, and because this lemma provides the final connection of the preceding theory with 2-HP. Thus, $\mathscr{X}$ will be given by $\{(x, y): 0 \leqq x \leqq a,\ 0 \leqq y \leqq b\}$, while $\mathscr{M}^\nu$ will be given by the lines $x = \xi_k^\nu$, $y = \eta_l^\nu$. Also, $f(x, y, V(x, y))$ will mean $f(x, y, V^0(x, y), V^1(x, y), V^2(x, y))$.

Lemma 3. *The components V^0, V^1, V^2 of V form a solution of the integral equations* (1.7).

Proof. Let

$$u(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y f(s, t, V(s, t))\, ds\, dt,$$

$$p(x, y) = \sigma'(x) + \int_0^y f(x, t, V(x, t))\, dt,$$

$$q(x, y) = \tau'(y) + \int_0^x f(s, y, V(s, y))\, ds.$$

Note that u, p and q are continuous (and therefore uniformly continuous) on $\mathscr{X}$. To prove the lemma it must be shown that $V^0 = u$, $V^1 = p$ and $V^2 = q$ on all of $\mathscr{X}$. Using the equation defining $V^{0\nu}$, that is, the first of equations (4.1), one finds

$$|V_{kl}^{0\nu} - u(\xi_k^\nu, \eta_l^\nu)| = \left| \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} C_{\alpha\beta}^\nu \Delta\xi_\alpha^\nu \Delta\eta_\beta^\nu - \int_0^{\xi_k^\nu} \int_0^{\eta_l^\nu} f(s, t, V(s, t))\, ds\, dt \right|$$

$$\leqq \left| \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} \left\{ \int_{\xi_\alpha^\nu}^{\xi_{\alpha+1}^\nu} \int_{\eta_\beta^\nu}^{\eta_{\beta+1}^\nu} C_{\alpha\beta}^\nu\, ds\, dt - \int_{\xi_\alpha^\nu}^{\xi_{\alpha+1}^\nu} \int_{\eta_\beta^\nu}^{\eta_{\beta+1}^\nu} f(s, t, V(s, t))\, ds\, dt \right\} \right|$$

$$\leqq \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} \int_{\xi_\alpha^\nu}^{\xi_{\alpha+1}^\nu} \int_{\eta_\beta^\nu}^{\eta_{\beta+1}^\nu} |C_{\alpha\beta}^\nu - f(s, t, V(s, t))|\, ds\, dt.$$

Because f and V are uniformly continuous on $\mathscr{X}$, given $\varepsilon > 0$, the mesh can be made fine enough, say $\nu > N_1$, so that (for $\xi_\alpha^\nu \leqq s < \xi_{\alpha+1}^\nu$, $\eta_\beta^\nu \leqq t < \eta_{\beta+1}^\nu$) if $\nu > N_1$, then $|C_{\alpha\beta}^\nu - f(s, t, V(s, t))| \leqq |C_{\alpha\beta}^\nu - f(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta})| + \varepsilon$. But the V^ν converge uniformly to V. Thus, if $\nu > N_2$, then, again because of the continuity of f,

$$|C_{\alpha\beta}^\nu - f(s, t, V(s, t))| \leqq |C_{\alpha\beta}^\nu - f(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta}^\nu)| + 2\varepsilon$$

which can be written, using the definition of $C_{\alpha\beta}^\nu$, as

$$|\mathscr{C}^\nu(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta}^\nu) - f(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta}^\nu)| + 2\varepsilon.$$

Further, because $\mathscr{C}^\nu$ converges uniformly to f, there is an N_3 such that if $\nu > N_3$, then

$$|\mathscr{C}^\nu(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta}^\nu) - f(\xi_\alpha^\nu, \eta_\beta^\nu, V_{\alpha\beta}^\nu)| < \varepsilon. \tag{4.4}$$

Hence

$$|V_{kl}^{0\nu} - u(\xi_k^\nu, \eta_l^\nu)| \leqq \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} 3\varepsilon \Delta\xi_\alpha^\nu \Delta\eta_\beta^\nu \leqq 3\varepsilon a b.$$

Since $V^{0\nu}$ converges uniformly to V^0 it follows that if $\nu > N_4$, then

$$|V_{kl}^0 - u(\xi_k^\nu, \eta_l^\nu)| \leqq |V_{kl}^0 - V_{kl}^{0\nu}| + |V_{kl}^{0\nu} - u(\xi_k^\nu, \eta_l^\nu)| \leqq \varepsilon + 3\varepsilon a b.$$

Finally, because V^0 and u are uniformly continuous, if the mesh $\mathscr{M}^\nu$ is fine enough, say $\nu > N_5$, so that for every (x, y) in $\mathscr{X}$ there is a node (ξ_k^ν, η_l^ν) for which $\|(x, y) - (\xi_k^\nu, \eta_l^\nu)\|$ is small, then

$$|V^0(x, y) - u(x, y)| \leqq 3\varepsilon + 3\varepsilon a b,$$

and hence $V^0 = u$ on all of $\mathscr{X}$.

Precisely parallel arguments hold for V^1 and V^2, so that $V^1 = p$ and $V^2 = q$ on all of $\mathscr{X}$. This completes the proof of the lemma.

For applications in the next section and for the Runge-Kutta procedure mentioned in the Introduction, the following is important.

Remark. In Lemma 2, for each ν, only a single function $\mathscr{C}^\nu(x, y, v)$ was used to define all three of $V^{0\nu}$, $V^{1\nu}$ and $V^{2\nu}$ (see eqn. (4.1)). In the proofs of Lemmas 2 and 3 the fundamental property of the sequence $\{\mathscr{C}^\nu\}$ was that of its uniform convergence to f (*cf.* the reasoning employed between (4.2) and (4.3) in Lemma 2, and the inequality (4.4) in Lemma 3). Instead of the single sequence, three sequences $\{\mathscr{C}^{0\nu}\}$, $\{\mathscr{C}^{1\nu}\}$ and $\{\mathscr{C}^{2\nu}\}$ could just as well be used, so that in equations (4.1) $C^{j\nu}$ would appear in the summation for $V^{j\nu}$. That the three functions $V^{j\nu}$ are well defined is easily checked, as before. The important requirement on the sequences $\{\mathscr{C}^{j\nu}\}$ (other than that each function is, of course, constant on each cell of $\mathscr{M}^{*\nu}$) is that they converge uniformly to f on $\mathscr{X} \times \mathscr{V}$. The key equations (4.3) and (4.4) then still hold.

The result of the preceding Lemmas 2 and 3, and the remark just made, may be summed up in the following theorem.

Theorem 3. *Let $\mathscr{X}$ be the set $\{(x, y): 0 \leqq x \leqq a,\ 0 \leqq y \leqq b\}$ in the real plane; let $\mathscr{V}$ be 3-dimensional real space with points (v^0, v^1, v^2). Let f be a real-valued function defined on $\mathscr{X} \times \mathscr{V}$ such that f is bounded and continuous and satisfies*

the Lipschitz condition, with Lipschitz constant L,

$$|f(x, y, v^0, v^1, v^2) - f(x, y, v^0, \bar{v}^1, \bar{v}^2)| \leqq L\{|v^1 - \bar{v}^1| + |v^2 - \bar{v}^2|\}$$

on all of $\mathscr{X} \times \mathscr{V}$. Let σ and τ be continuously differentiable real-valued functions defined on $0 \leqq x \leqq a$ and $0 \leqq y \leqq b$ respectively, and suppose $\sigma(0) = \tau(0)$.

Let $\{\mathscr{M}^\nu\}$ and $\{\mathscr{M}^{*\nu}\}$ be convergent sequences of meshes in $\mathscr{X}$ and $\mathscr{X} \times \mathscr{V}$, respectively, and $\{\mathscr{C}^{j\nu}\}$ $(j = 0, 1, 2)$ sequences of real-valued functions defined on $\mathscr{X} \times \mathscr{V}$ such that $\mathscr{C}^{j\nu}$ is constant on each cell of $\mathscr{M}^{*\nu}$, and such that $\{\mathscr{C}^{j\nu}\}$ converges uniformly to f as $\nu \to \infty$ (j fixed).

Let $\mathscr{N}^\nu$ denote the set of nodes of $\mathscr{M}^\nu$.

Then, a sequence of functions V^ν: $\mathscr{N}^\nu \to \mathscr{V}$ is defined by equations (4.1) with the property that a subsequence converges uniformly to a continuous function V: $\mathscr{X} \to \mathscr{V}$. The first component V^0 of V is a solution of the differential equation

$$u_{xy} = f(x, y, u, u_x, u_y)$$

on $\mathscr{X}$ and satisfies the characteristic initial conditions

$$u(x, 0) = \sigma(x) \quad \text{and} \quad u(0, y) = \tau(y).$$

The second and third components V^1 and V^2 are, respectively, the x and y derivatives of the solution V^0.

§ 5. Convergence to a Given Solution

It is well known that if the Lipschitz requirement on f in Theorem 3 is replaced by the stronger requirement that

$$|f(x, y, v^0, v^1, v^2) - f(x, y, \bar{v}^0, \bar{v}^1, \bar{v}^2)| \leq L\{|v^0 - \bar{v}^0| + |v^1 - \bar{v}^1| + |v^2 - \bar{v}^2|\},$$

then the solution to 2-HP is unique, while under the weaker condition in Theorem 3 there may be several solutions. (A simple illustration is given, for example, by DIAZ [*3*], p. 361.) It is of interest to note that every continuously differentiable solution can be obtained as the limit of a sequence of approximate solutions of the type described in the preceding sections. This was shown by ZWIRNER [*10*] for the case when $f = f(x, y, u)$.

Theorem 4. *Let $\mathscr{X}$, $\mathscr{V}$, f, σ and τ be as in the hypothesis of Theorem 3. Let $u(x, y)$ be an arbitrary solution on $\mathscr{X}$ of*

$$u_{xy} = f(x, y, u, u_x, u_y), \quad u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y), \tag{5.1}$$

where u, u_x and u_y are continuous on $\mathscr{X}$.

Then there is a sequence of vines of the type considered in Lemma 2 such that this sequence converges uniformly in $\mathscr{X}$ to a function V: $\mathscr{X} \to \mathscr{V}$, and the first component of V is $u(x, y)$.

Proof. Let $\{\mathscr{M}^\nu\}$ and $\{\mathscr{M}^{*\nu}\}$ be convergent sequences of meshes on $\mathscr{X}$ and $\mathscr{X} \times \mathscr{V}$ respectively, defined by $x = \xi_k^\nu$, $y = \eta_l^\nu$, $v^j = \omega_r^{j\nu}$ $(j = 0, 1, 2)$. In general terms, all that is needed to prove the theorem is, for each ν, to define "suitable" functions $\mathscr{C}^{0\nu}$, $\mathscr{C}^{1\nu}$ and $\mathscr{C}^{2\nu}$ to be used respectively in the three equations (4.1). To this end, let $p = u_x$, $q = u_y$, $u_{\alpha\beta} = u(\xi_\alpha^\nu, \eta_\beta^\nu)$, $p_{\alpha\beta} = p(\xi_\alpha^\nu, \eta_\beta^\nu)$, $q_{\alpha\beta} = q(\xi_\alpha^\nu, \eta_\beta^\nu)$

and define

$$C^{0\nu}_{\alpha\beta}=\frac{u_{\alpha+1,\beta+1}-u_{\alpha+1,\beta}-u_{\alpha,\beta+1}+u_{\alpha\beta}}{\Delta\xi^\nu_\alpha\,\Delta\eta^\nu_\beta}, \tag{5.2}$$

$$C^{1\nu}_{\alpha\beta}=\frac{p_{\alpha+1,\beta}-p_{\alpha\beta}}{\Delta\eta^\nu_\beta}, \tag{5.3}$$

$$C^{2\nu}_{\alpha\beta}=\frac{q_{\alpha,\beta+1}-q_{\alpha\beta}}{\Delta\xi^\nu_\alpha}. \tag{5.4}$$

To define $\mathscr{C}^{j\nu}$, on each cell of $\mathscr{M}^{*\nu}$ containing points $(x,y,u(x,y),p(x,y),q(x,y))$ for which $\xi^\nu_\alpha\leqq x<\xi^\nu_{\alpha+1}$ and $\eta^\nu_\beta\leqq y<\eta^\nu_{\beta+1}$, let the value of $\mathscr{C}^{j\nu}$ be $C^{j\nu}_{\alpha\beta}$. On each remaining cell of $\mathscr{M}^{*\nu}$ let the value of $\mathscr{C}^{j\nu}$ be any value taken on by f in that cell.

From the ordinary mean value theorem of differential calculus, it is evident that

$$C^{1\nu}_{\alpha\beta}=\frac{\partial}{\partial y}p(\xi^\nu_\alpha,\bar\eta^\nu_\beta)\quad\text{and}\quad C^{2\nu}_{\alpha\beta}=\frac{\partial}{\partial x}q(\bar\xi^\nu_\alpha,\eta^\nu_\beta)$$

where

$$\eta^\nu_\beta<\bar\eta^\nu_\beta<\eta^\nu_{\beta+1}\quad\text{and}\quad\xi^\nu_\alpha<\bar\xi^\nu_\alpha<\xi^\nu_{\alpha+1}.$$

From a similar mean value theorem for functions of two independent variables (see GOURSAT [4]) it follows that

$$C^{0\nu}_{\alpha\beta}=\frac{\partial^2}{\partial x\,\partial y}u(\xi^{*\nu}_\alpha,\eta^{*\nu}_\beta),\qquad\text{where }\xi^\nu_\alpha<\xi^{*\nu}_\alpha<\xi^\nu_{\alpha+1},\quad\eta^\nu_\beta<\eta^{*\nu}_\beta<\eta^\nu_{\beta+1}.$$

Thus, using the differential equation (5.1) (note that $p_y=q_x=u_{xy}$), one finds

$$\begin{aligned}
C^{0\nu}_{\alpha\beta}&=f\left(\xi^{*\nu}_\alpha,\eta^{*\nu}_\beta,u(\xi^{*\nu}_\alpha,\eta^{*\nu}_\beta),\ p(\xi^{*\nu}_\alpha,\eta^{*\nu}_\beta),\ q(\xi^{*\nu}_\alpha,\eta^{*\nu}_\beta)\right),\\
C^{1\nu}_{\alpha\beta}&=f\left(\xi^\nu_\alpha,\bar\eta^\nu_\beta,\ u(\xi^\nu_\alpha,\bar\eta^\nu_\beta),\ p(\xi^\nu_\alpha,\bar\eta^\nu_\beta),\ q(\xi^\nu_\alpha,\bar\eta^\nu_\beta)\right),\\
C^{2\nu}_{\alpha\beta}&=f\left(\bar\xi^\nu_\alpha,\eta^\nu_\beta,\ u(\bar\xi^\nu_\alpha,\eta^\nu_\beta),\ p(\bar\xi^\nu_\alpha,\eta^\nu_\beta),\ q(\bar\xi^\nu_\alpha,\eta^\nu_\beta)\right).
\end{aligned}$$

From the continuity of f,u,p and q it is thus clear that the sequences $\{\mathscr{C}^{j\nu}\}$ $(j=0,1,2)$ converge to f as $\nu\to\infty$.

It remains only to be shown that the functions $V^{0\nu}$, $V^{1\nu}$ and $V^{2\nu}$ defined by equations (4.1) using the above functions $\mathscr{C}^{0\nu}$, $\mathscr{C}^{1\nu}$ and $\mathscr{C}^{2\nu}$ respectively, do converge to u, p and q. In fact, it will be shown that $V^{0\nu}_{kl}=u(\xi^\nu_k,\eta^\nu_l)$, $V^{1\nu}_{kl}=p(\xi^\nu_k,\eta^\nu_l)$ and $V^{2\nu}_{kl}=q(\xi^\nu_k,\eta^\nu_l)$. The last two of these equalities are established immediately upon the substitution of the formulas (5.3) and (5.4) for $C^{1\nu}_{\alpha\beta}$ and $C^{2\nu}_{\alpha\beta}$ into the last two of equations (4.1) respectively. It is a little more difficult to derive the first equality. Formula (5.2) for $C^{0\nu}_{\alpha\beta}$ substituted in the first of equations (4.1) yields the result (in the present notation)

$$V^{0\nu}_{kl}=\sigma_k+\tau_l-\sigma_0+\sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{l-1}(u_{\alpha+1,\beta+1}-u_{\alpha+1,\beta}-u_{\alpha,\beta+1}+u_{\alpha,\beta}). \tag{5.5}$$

In this,

$$\begin{aligned}
\sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{l-1}u_{\alpha+1,\beta+1}&=\sum_{\alpha=1}^{k}\sum_{\beta=1}^{l}u_{\alpha\beta}=\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1}u_{\alpha\beta}+\sum_{\alpha=1}^{k-1}u_{\alpha l}+\sum_{\beta=1}^{l-1}u_{k\beta}+u_{kl},\\
\sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{l-1}u_{\alpha+1,\beta}&=\sum_{\alpha=1}^{k}\sum_{\beta=0}^{l-1}u_{\alpha\beta}=\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1}u_{\alpha\beta}+\sum_{\alpha=1}^{k-1}u_{\alpha 0}+\sum_{\beta=1}^{l-1}u_{k\beta}+u_{k0},\\
\sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{l-1}u_{\alpha,\beta+1}&=\sum_{\alpha=0}^{k-1}\sum_{\beta=1}^{l}u_{\alpha\beta}=\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1}u_{\alpha\beta}+\sum_{\alpha=1}^{k-1}u_{\alpha l}+\sum_{\beta=1}^{l-1}u_{0\beta}+u_{0l},\\
\sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{l-1}u_{\alpha\beta}&=\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1}u_{\alpha\beta}+\sum_{\alpha=1}^{k-1}u_{\alpha 0}+\sum_{\beta=1}^{l-1}u_{0\beta}+u_{00}.
\end{aligned}$$

The expressions on the right may now be substituted for the appropriate summations in (5.5). All the terms but u_{kl} then cancel on the right of (5.5) since $u_{k0}=\sigma_k$, $u_{0l}=\tau_l$ and $u_{00}=\sigma_0$. Thus $V_{kl}^{0\nu}=u_{kl}$ as asserted.

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KIRCHGÄSSNER, K., Beiträge zu einer nichtlinearen Theorie der Stabilität von Schichtenströmungen längs zylindrisch gekrümmter Wände gegenüber dreidimensionalen Störungen 20

STERNBERG, E., On the Integration of the Equations of Motion in the Classical Theory of Elasticity 34

BERNSTEIN, B., & R. A. TOUPIN, Korn Inequalities for the Sphere and Circle 51

LEVIN, J. J., On the Global Asymptotic Behavior of Nonlinear Systems of Differential Equations 65

MOORE, R. H., On Approximate Solutions of Non-linear Hyperbolic Partial Differential Equations 75

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